Hybrid high-order (HHO) method for modeling and numerical simulation of seismic-acoustic wave propagation

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Context and issues

Objectives

- Accurate modelisation and simulation of seismo-acoustic waves **through heterogeneous domains with complex geometries**
- Treatment of realistic cases of interest
  - **High Performance Computing (HPC)**

Issues

- Difficulty to mesh complex geometries
- High-order precision needed to accurately capture waves
  - **Hybrid discontinuous methods (HDG/HHO)**

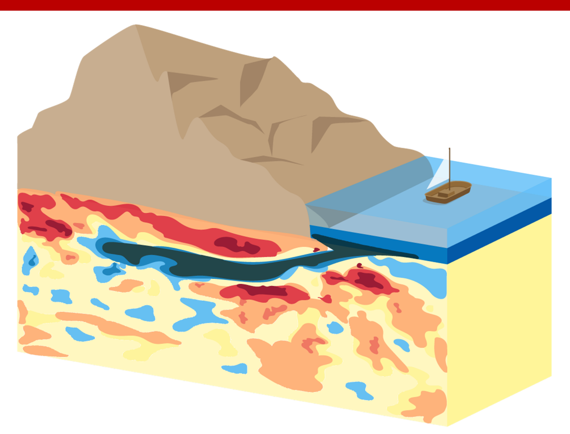

Fig. 1: Lateral heterogeneities near the earth's surface.

Coupling of the acoustic and elastic wave equations

- **Acoustic wave equation:** $\begin{cases} \rho_{\rm F}\partial_t \boldsymbol{v}^{\rm F}(t) + \nabla p(t) = \boldsymbol{0} \\ \frac{1}{\kappa}\partial_t p(t) + \nabla\cdot\boldsymbol{v}^{\rm F}(t) = g(t) \end{cases}$
- **Elastic wave equation:** $\begin{cases} \partial_t \boldsymbol{\varepsilon}(t) - \nabla^s \boldsymbol{v}^{\rm S}(t) = \boldsymbol{0} \\ \rho_{\rm S}\partial_t \boldsymbol{v}^{\rm S}(t) - \nabla\cdot(\boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}(t)) = \boldsymbol{f}(t) \end{cases}$
- **Coupling condition:** $\begin{cases} [\![\boldsymbol{v}(t)\cdot\boldsymbol{n}_\Gamma]\!]_\Gamma = 0 \\ (\boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}(t))\cdot\boldsymbol{n}_\Gamma = p(t)\boldsymbol{n}_\Gamma \end{cases}$

The HHO method

Degrees of freedom

- **Principle:** Polynomial unknowns located in the cells and on the faces

HHO unknowns:

$$\hat{u}_h := (u_{\mathcal{T}}, u_{\mathcal{F}})$$

Fig. 2: Left panel: Equal-order discretization ($k' = k = 0$). Right panel: Mixed-order discretization ($k' = k + 1 = 1$).

Operators

- **Gradient reconstruction operator:** $\nabla u \;\rightarrow\; \mathrm{G}(\hat{u}_h)$
- **Stabilization operator:** $\mathrm{s}(\hat{u}_h, \hat{w}_h)$
  - Penalization at the element level to ensure stability while preserving the approximation properties of the reconstruction.

Advantages

- **Mesh flexibility:**
  - Complex geometries
  - Unstructured and polyhedral meshes
  - Local mesh refinement
- **Local conservativity**
- **Optimal error estimates for smooth solutions**
- **Attractive computational costs:**
  - Global problem couples only face dofs
  - Cell dofs recovered by local post-processing

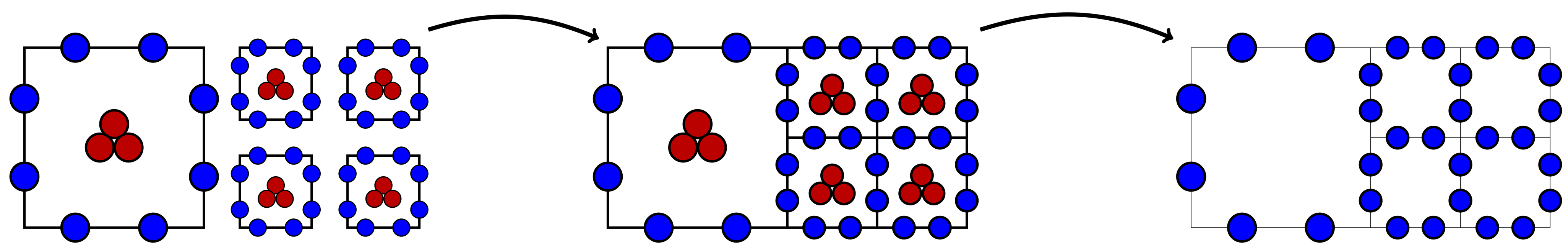

Fig. 3: Static condensation procedure.

HHO space semi-discretization

- **Approximation spaces:**
  - **Acoustic domain:**

$$\boldsymbol{V}^k_{\mathcal{T}_{\rm F}} := \bigtimes_{T\in\mathcal{T}_h^{\rm F}} \mathbb{P}^k\left(T;\mathbb{R}^d\right), \qquad \hat{V}_h^{\rm F} := \bigtimes_{T\in\mathcal{T}_h^{F}} \mathbb{P}^{k'}(T;\mathbb{R}) \times \bigtimes_{F\in\mathcal{F}_h^{\rm F}} \mathbb{P}^k(F;\mathbb{R})$$

  - **Elastic domain:**

$$\boldsymbol{\mathcal{Z}}^k_{\mathcal{T}_{\rm S}} := \bigtimes_{T\in\mathcal{T}_h^{\rm S}} \mathbb{P}^k\left(T;\mathbb{R}^{d\times d}_{\rm sym}\right), \quad \hat{\boldsymbol{V}}_h^{\rm S} := \bigtimes_{T\in\mathcal{T}_h^{\rm S}} \mathbb{P}^{k'}\left(T;\mathbb{R}^d\right) \times \bigtimes_{F\in\mathcal{F}_h^{\rm S}} \mathbb{P}^k\left(F;\mathbb{R}^d\right)$$

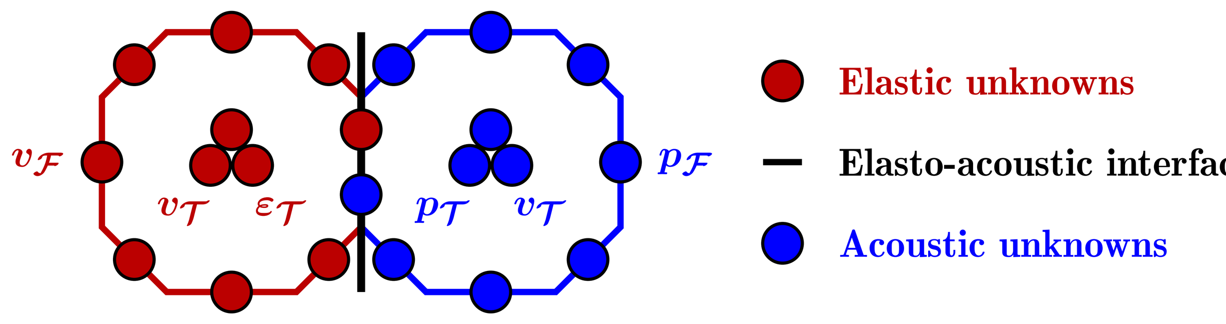

Fig. 4: Elasto-acoustic unknowns with a mixed-order ($k' = k + 1 = 1$) discretization.

- **Elasto-acoustic coupling:**

$$\begin{cases} (\partial_t \boldsymbol{v}^{\rm F}_{\mathcal{T}}(t), \boldsymbol{r}_{\mathcal{T}})_{\boldsymbol{L}^2(\rho_{\rm F};\Omega_{\rm F})} + (\boldsymbol{G}_{\mathcal{T}}(\hat{p}_h(t)), \boldsymbol{r}_{\mathcal{T}})_{\boldsymbol{L}^2(\Omega_{\rm F})} = 0 \\ (\partial_t p_{\mathcal{T}}(t), q_{\mathcal{T}})_{L^2(\frac{1}{\kappa};\Omega_{\rm F})} - (\boldsymbol{v}^{\rm F}_{\mathcal{T}}(t), \boldsymbol{G}_{\mathcal{T}}(\hat{q}_h))_{\boldsymbol{L}^2(\Omega_{\rm F})} + s^{\rm F}_h(\hat{p}_h(t), \hat{q}_h) - (\boldsymbol{v}^{\rm S}_{\mathcal{F}}(t)\cdot\boldsymbol{n}_\Gamma, q_{\mathcal{F}})_{L^2(\Gamma)} = (g(t), q_{\mathcal{T}})_{L^2(\Omega_{\rm F})} \\ (\partial_t \boldsymbol{\varepsilon}_{\mathcal{T}}(t), \boldsymbol{z}_{\mathcal{T}})_{\boldsymbol{L}^2(\Omega_{\rm S})} - (\boldsymbol{E}_{\mathcal{T}}(\hat{\boldsymbol{v}}_h(t)), \boldsymbol{z}_{\mathcal{T}})_{\boldsymbol{L}^2(\Omega_{\rm S})} = 0 \\ (\partial_t \boldsymbol{v}_{\mathcal{T}}(t), \boldsymbol{w}_{\mathcal{T}})_{\boldsymbol{L}^2(\rho;\Omega_{\rm S})} + (\boldsymbol{\varepsilon}_{\mathcal{T}}, \boldsymbol{E}_{\mathcal{T}}(\hat{\boldsymbol{w}}_h))_{\boldsymbol{L}^2(\mathcal{C}:\Omega_{\rm S})} + s^{\rm S}_h(\hat{\boldsymbol{v}}^{\rm S}_h(t), \hat{\boldsymbol{w}}_h) + (p_{\mathcal{F}}(t), \boldsymbol{w}_{\mathcal{F}}\cdot\boldsymbol{n}_\Gamma)_{L^2(\Gamma)} = (\boldsymbol{f}(t), \boldsymbol{w}_{\mathcal{T}})_{\boldsymbol{L}^2(\Omega_{\rm S})} \end{cases}$$

- **Algebraic realization:**

$$\begin{bmatrix} \mathrm{M}^{\rm v}_{\mathcal{TT}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{M}^{\rm F}_{\mathcal{TT}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{M}^{\varepsilon}_{\mathcal{TT}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{M}^{\rm S}_{\mathcal{TT}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_t \mathrm{V}^{\rm F}_{\mathcal{T}} \\ \partial_t \mathrm{P}_{\mathcal{T}} \\ \partial_t \mathrm{P}_{\mathcal{F}} \\ \partial_t \mathrm{S}_{\mathcal{T}} \\ \partial_t \mathrm{V}_{\mathcal{T}} \\ \partial_t \mathrm{V}_{\mathcal{F}} \end{bmatrix} + \begin{bmatrix} 0 & -\mathrm{G}_{\mathcal{T}} & -\mathrm{G}_{\mathcal{F}} & 0 & 0 & 0 \\ \mathrm{G}^\dagger_{\mathcal{T}} & \Sigma^{\rm F}_{\mathcal{TT}} & \Sigma^{\rm F}_{\mathcal{TF}} & 0 & 0 & 0 \\ \mathrm{G}^{\mathcal{F}}_{\mathcal{F}} & \Sigma^{\rm F}_{\mathcal{FT}} & \Sigma^{\rm F}_{\mathcal{FF}} & 0 & 0 & \mathrm{C}_\Gamma \\ 0 & 0 & 0 & 0 & -\mathrm{E}_{\mathcal{T}} & -\mathrm{E}_{\mathcal{F}} \\ 0 & 0 & 0 & \mathrm{E}^\dagger_{\mathcal{T}} & \Sigma^{\rm S}_{\mathcal{TT}} & \Sigma^{\rm S}_{\mathcal{TF}} \\ 0 & 0 & -\mathrm{C}^\dagger_\Gamma & \mathrm{E}^\dagger_{\mathcal{F}} & \Sigma^{\rm S}_{\mathcal{FT}} & \Sigma^{\rm S}_{\mathcal{FF}} \end{bmatrix} \begin{bmatrix} \mathrm{V}^{\rm F}_{\mathcal{T}} \\ \mathrm{P}_{\mathcal{T}} \\ \mathrm{P}_{\mathcal{F}} \\ \mathrm{S}_{\mathcal{T}} \\ \mathrm{V}^{\rm S}_{\mathcal{T}} \\ \mathrm{V}^{\rm S}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{G}_{\mathcal{T}} \\ 0 \\ 0 \\ \mathrm{F}_{\mathcal{T}} \\ 0 \end{bmatrix}$$

Numerical results

- **Verification of convergence rates on analytical solutions:**
  - $\mathcal{O}(h^{k+1})$ in H^1-norm
  - $\mathcal{O}(h^{k+2})$ in L^2-norm **(superconvergence)**

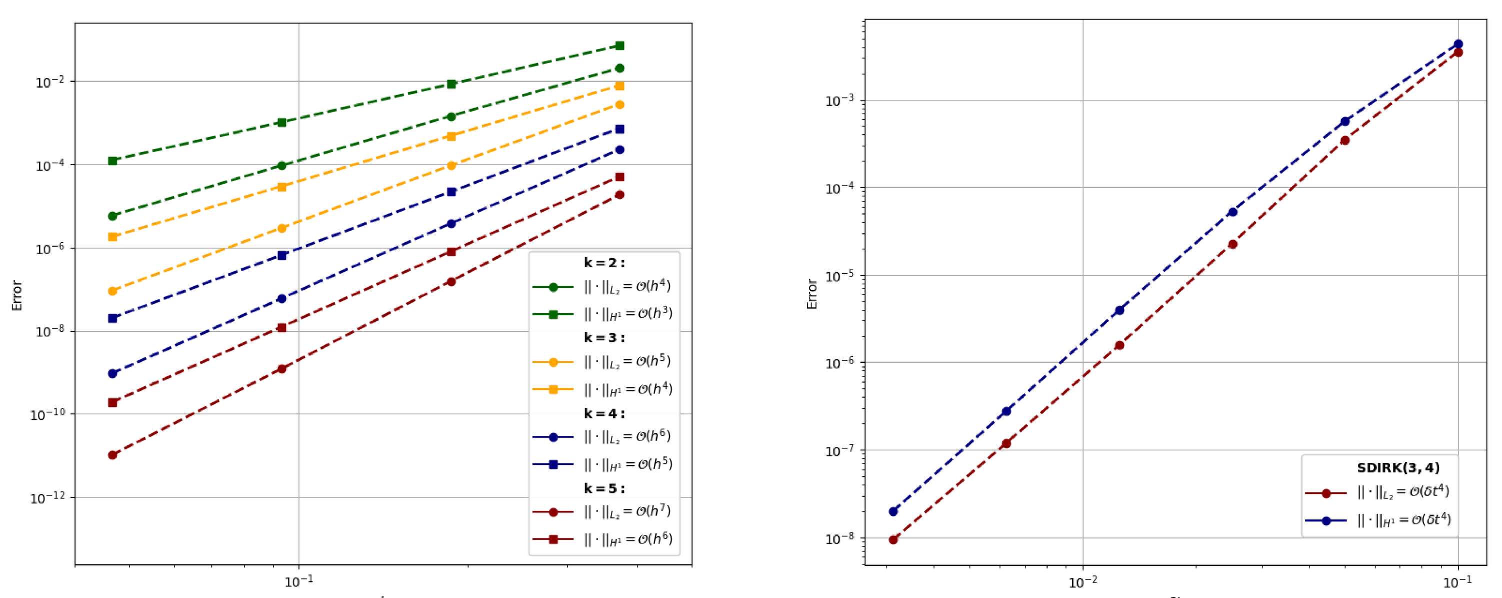

Fig. 5: Left panel: Errors as a function of the mesh size with $\Delta t = 0.1 \times 2^{-5}$. Right panel: Errors as a function of the time-step with $k' = k + 1 = 6$ and $\mathrm{dx} = 2^{-5}$.

- **Realistic test case**
  - **Computational domain:**
    - Acoustic region on the upper side
    - Elastic region on the lower side
  - **Homogeneous Dirichlet conditions**
  - **Intial condition:** pressure Ricker wavelet

$$p_0(x,y) := -\frac{4}{10}\sqrt{\frac{10}{3}}(1600r^2 - 1)\pi^{-1/4}\, e^{-800r^2}$$

$$\boldsymbol{v}^{\rm F}_0 := \boldsymbol{0}, \qquad \boldsymbol{v}^{\rm S}_0 := \boldsymbol{0}, \qquad \boldsymbol{\varepsilon}_0 := \boldsymbol{0}.$$

  - **Time integration scheme:** SDIRK(s,s+1)

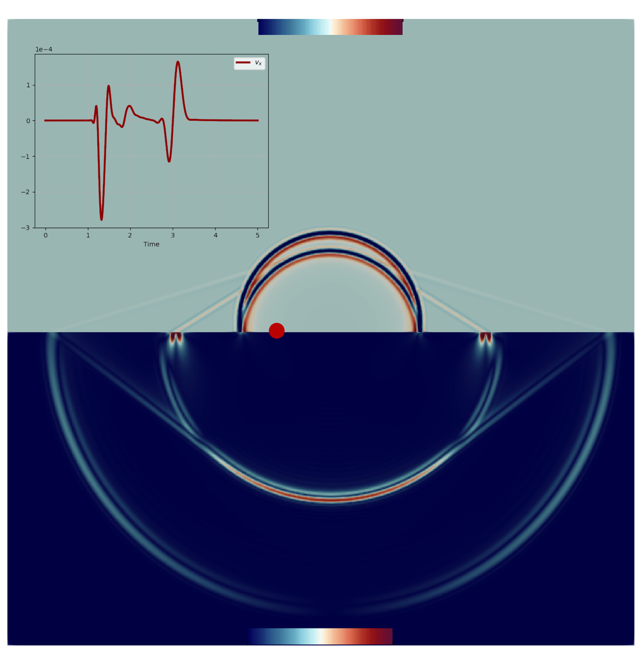

Fig. 6: Two-dimensional distribution of the acoustic pressure (upper side) and elastic velocity norm (lower side), predicted by the HHO-SDIRK (3, 4) at $t = 5$ s. Simulation parameters: $k' = k + 1 = 2, \mathrm{dx} = 2^{-8}$ and $\Delta t = 0.1 \times 2^{-8}$.

Energy conservation of the scheme

- **Mechanical energy of the scheme:** $\mathcal{E}_h(t) := \mathcal{E}^{\rm S}_h(t) + \mathcal{E}^{\rm F}_h(t)$ with

$$\mathcal{E}^{\rm F}_h(t) := \frac{1}{2}||\boldsymbol{v}^{\mathbf{F}}_{\mathcal{T}}(t)||^2_{\boldsymbol{L}^2(\rho_{\rm F};\Omega_{\rm F})} + \frac{1}{2}||p_{\mathcal{T}}(t)||^2_{L^2(\frac{1}{\kappa};\Omega_{\rm F})}, \qquad \mathcal{E}^{\mathbf{S}}_h(t) := \frac{1}{2}||\boldsymbol{v}^{\mathbf{S}}_{\mathcal{T}}(t)||^2_{\boldsymbol{L}^2(\rho_{\rm S};\Omega_{\rm S})} + \frac{1}{2}||\boldsymbol{\varepsilon}_{\mathcal{T}}(t)||^2_{\boldsymbol{L}^2(\mathcal{C};\Omega_{\rm S})}$$

- **Semi-discrete energy conservation of the scheme**

$$\mathcal{E}_h(t) = \mathcal{E}_h(0) + \int_0^t \left[(\boldsymbol{f}(\alpha), \boldsymbol{v}^{\rm S}_{\mathcal{T}}(\alpha))_{\boldsymbol{L}^2(\Omega_{\rm S})} + (g(\alpha), p_{\mathcal{T}}(\alpha))_{L^2(\Omega_{\rm F})} - s^{\rm S}_h(\hat{\boldsymbol{v}}^{\rm S}_h(\alpha), \hat{\boldsymbol{v}}^{\rm S}_h(\alpha)) - s^{\rm F}_h(\hat{p}_h(\alpha), \hat{p}_h(\alpha))\right] \mathrm{d}\alpha$$

- **Validation on analytic test cases**

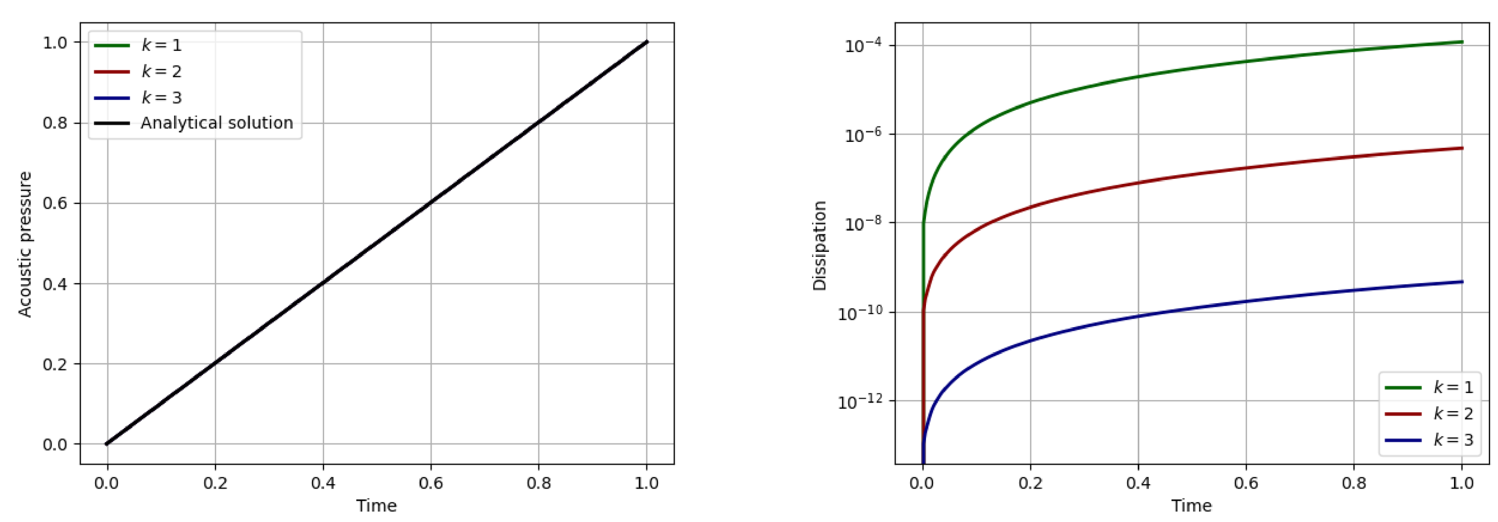

Fig. 7: Demonstration of the negligible nature of the energy dissipation introduced by the HHO scheme.

Some references

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