Unfitted HHO method with polynomial extension for elliptic interface problems

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Domain decomposition

- **Domain Ω:** $\overline{\Omega} := \overline{\Omega_1} \cup \overline{\Omega_2}$
- **Interface Γ:** $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$
- **Jump across Γ:** $[\![u]\!]_\Gamma := u_{|\Omega_1} - u_{|\Omega_2}$

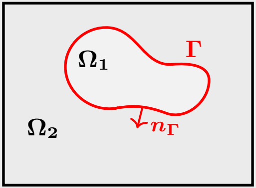

Fig. 1: Model problem

Elliptic interface problem

- **Strong form:** Find $u \in H^1(\Omega_1 \cup \Omega_2)$ such that

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = f & \text{in } \Omega_1 \cup \Omega_2 \\ [\![u]\!]_\Gamma = g_D & \text{on } \Gamma \\ [\![\kappa \nabla u]\!]_\Gamma \cdot \boldsymbol{n}_\Gamma = g_N & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Unfitted methods

- **Minimize complexity of mesh generation**
- **Handle cut cells by doubling unknowns**
- **Need to integrate polynomials in cut cells** (e.g. by submeshing)
- **Price to pay :** **Need to stabilize ill-cut cells**

Unfitted FEM methods

- Introduced by [Hansbo and Hansbo, 2002]
- Standart technique for stabilization: **Ghost penalty** [Burman, 2010]

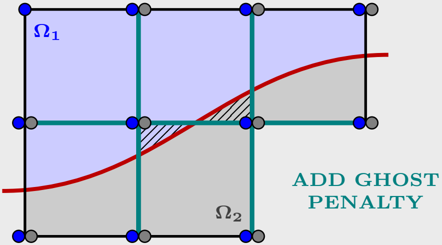

Fig. 2: Doubling of Q_1-FEM unknowns, ill-cut cells (dashes) and set of ghost-penalty faces

Fitted HHO methods

- **Seminal papers:** [Di Pietro, Ern, and Lemaire, 2014], [Di Pietro and Ern, 2015]
- **Main features:**
  - Design based on cell and face unknowns
  - General meshes: polyhedral meshes, hanging nodes
  - Attractive computational cost: Static condensation
  - Local conservativity at the cell level

Unfitted HHO methods

- **Seminal papers:** [Burman and Ern, 2018] [Burman, Cicuttin, Delay, and Ern, 2021]
- **Main features:**
  - Doubling of cell and face unknowns in cut cells
  - Cut stabilization by cell agglomeration
- **New approach for cut stabilization: Polynomial extension**
  - Use of similar technique for unfitted FEM [Badia, Verdugo, and Martín, 2018]

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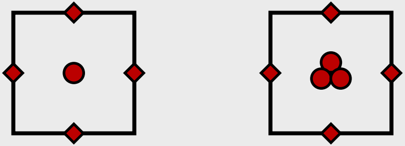
Degrees of freedom

- **Polynomial unknowns attached to mesh cells and faces**



HHO unknowns:

$$\hat{u}_h := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \hat{\mathcal{U}}_h$$

● **Cell unknowns, degree $k' \in \{k, k+1\}$** ◆ **Face unknowns, degree $k \geq 0$**

Fig. 3: Local HHO unknowns. **Left:** $k' = k = 0$. **Right:** $k' = k + 1 = 1$.

- Equal-order: $k' = k$
- Mixed-order: $k' = k + 1$

Global degrees of freedom

- Mesh $\mathcal{T}_h$ with faces $\mathcal{F}_h$
- **Global HHO spaces:** $\widehat{\mathcal{U}}_h := \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^{k'}(T;\mathbb{R}) \times \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^{k}(F;\mathbb{R})$

Design of the local gradient reconstruction operator

- **Gradient reconstruction operator:**
  - $(\boldsymbol{\nabla u})_{|T} \rightarrow \mathbf{G}_{\boldsymbol{T}}(\hat{\boldsymbol{u}}_T) \in \mathbb{P}^k(T;\mathbb{R}^d)$

    Design of $\mathbf{G}_{\boldsymbol{T}}(\hat{\boldsymbol{u}}_T)$ mimics an integration by parts

$$(\boldsymbol{G}_T(\hat{u}_T), \boldsymbol{q})_T = (\nabla u_T, \boldsymbol{q})_T - (u_T - u_{\partial T}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{\partial T}, \quad \forall \boldsymbol{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$$

Design of the local stabilization operator

- **Stabilization operator:** $\boldsymbol{\delta}_{\partial T}(\hat{\boldsymbol{u}}_T) := \boldsymbol{u}_{\partial T} - \boldsymbol{u}_{T|\partial T} \approx \mathbf{0}$

  Matching of cell dofs trace with face dofs (weakly)

  - **Equal-order discretization:** Specific stabilization to HHO (not used in unfitted HHO)
  - **Mixed-order discretization:** Same as HDG (Lehrenfeld-Schöberl)

$$s_T(\hat{u}_T, \hat{w}_T) := \kappa\, h_T^{-1} (\Pi^k_{\partial T}(u_T - u_{\partial T}), w_T - w_{\partial T})_{\partial T}$$

Main advantages of HHO methods

- **Improved error estimates for smooth solutions:**
  - H^1-error: $\mathcal{O}(h^{k+1})$
  - L^2-error: $\mathcal{O}(h^{k+2})$
- **Attractive computational costs:** Elimination of cell unknowns by Schur complement (static condensation) :
  - Global problem couples only face dofs
  - Cell dofs recovered by local post-processing

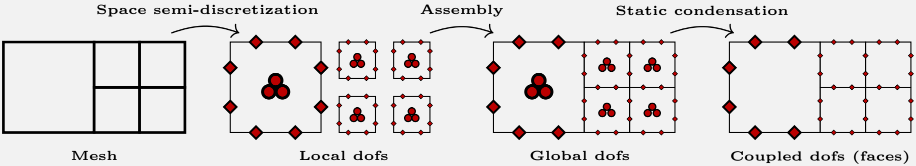


Fig. 4: Assembly and Schur complement procedure in the framework of HHO schemes

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Unfitted meshes and local unknowns

- **Mesh partitioning:** $\mathcal{T}_h := \mathcal{T}_h^{\circ} \cup \mathcal{T}_h^{\mathrm{OK}} \cup \mathcal{T}_h^{\mathrm{KO}}$
  - $\mathcal{T}_h^{\mathrm{KO},1} \cup \mathcal{T}_h^{\mathrm{KO},2} = \emptyset$ if mesh fine enough [Burman and Ern, 2018]
- **Doubling local unknowns in cut cells:**

$$\hat{u}_T := (\hat{u}_{T^1}, \hat{u}_{T^2}) := (u_{T^1}, u_{(\partial T)^1}, u_{T^2}, u_{(\partial T)^2}) \in \hat{\mathcal{U}}_T := \hat{\mathcal{U}}_{T^1} \times \hat{\mathcal{U}}_{T^2}$$

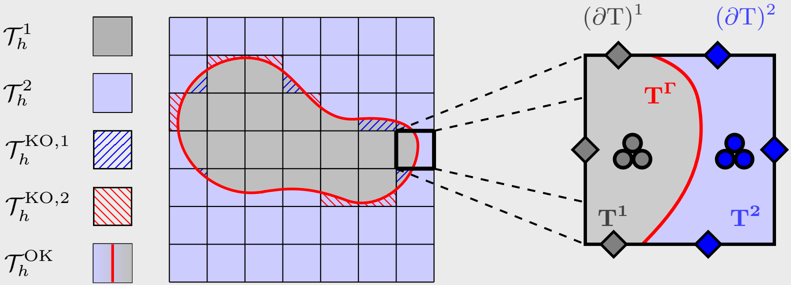

Fig. 5: Left. Types of cells involved in unfitted meshes.
Right. Local dofs in cut cell.

Pairing operator

$$\mathcal{N}_i : \mathcal{T}_h^{\mathrm{KO},i} \ni S \longmapsto T \in (\mathcal{T}_h^i \cup \mathcal{T}_h^{\mathrm{OK}} \cup \mathcal{T}_h^{\mathrm{KO},\bar{i}}) \cap \Delta_1(S), \quad \forall i \in \{1,2\}$$

- $\Delta_1(S)$: first layer of neighboring cells of S

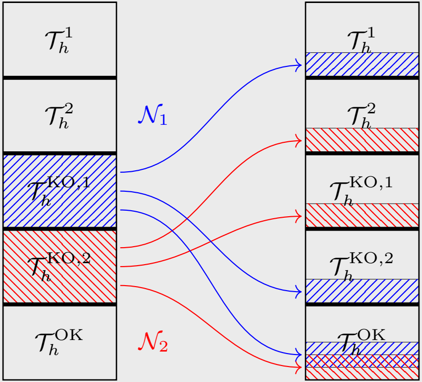

Fig. 6: Pairing of ill-cut cells

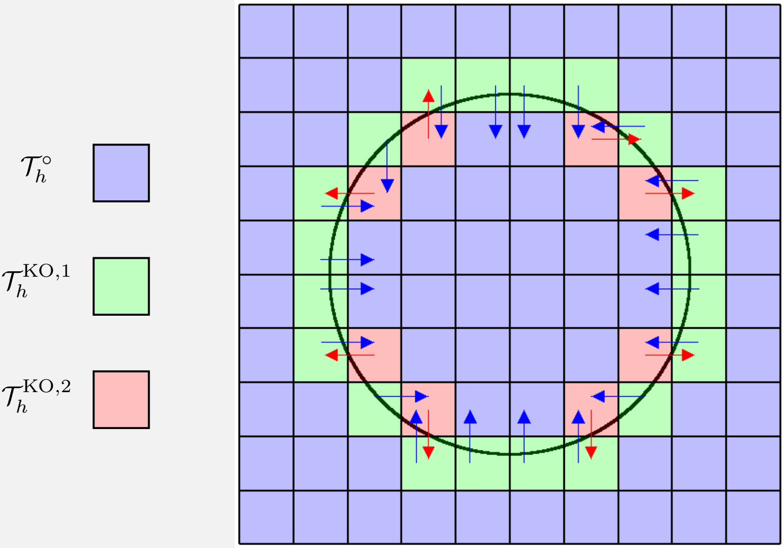

Fig. 7: Exemple of pairing procedure for coarse Cartesian mesh cut by circular interface

Cell agglomeration vs. Polynomial extension

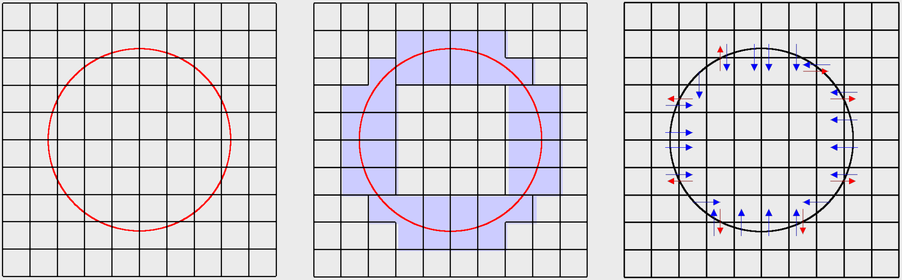

Fig. 8: **Left.** Initial mesh with circular interface. **Middle.** Cell agglomeration. **Right.** Stencil modification for polynomial extension

- **Cell agglomeration:**
  - ✓ Leverages on polyhedral capacity of HHO methods
  - ✗ Intrusive on mesh data structure
- **Polynomial extension:**
  - ✓ Works on initial mesh (non-intrusive)
  - ✗ Requires modification of the stencil (intrusive at the assembly level)

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- **UNCUT CELLS:** $T \in \mathcal{T}_h^i$
  - Stencil includes dofs of ill-cut cell(s)

$$\hat{u}_T^+ := (\hat{u}_T, (\hat{u}_S)_{S\in\mathcal{N}^{-1}(T)})$$

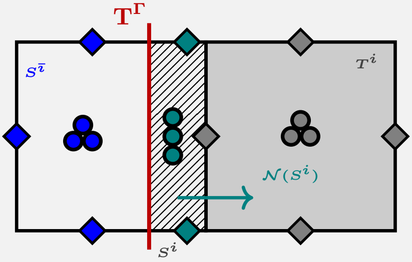

Fig. 9: Pairing configuration

Design of the local gradient reconstruction in the uncut cells

- **Classical gradient reconstruction:**

$$(\boldsymbol{G}_T(\hat{u}_T), \boldsymbol{q})_T = (\nabla u_T, \boldsymbol{q})_T - (u_T - u_{\partial T}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{\partial T}$$

- **Gradient reconstruction with polynomial extension:**

$$(\boldsymbol{G}_T^k(\hat{u}_T^+), \boldsymbol{q})_T := (\nabla u_T, \boldsymbol{q})_T - (u_T - u_{\partial T}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{\partial T}$$
$$+ \sum_{S\in\mathcal{N}_i^{-1}(T)} \Big\{ (\nabla u_T, \boldsymbol{q})_{S^i} - (u_T - u_{(\partial S)^i}, \boldsymbol{q}\cdot\boldsymbol{n}_S)_{(\partial S)^i} - \delta_{i1}\kappa_1 (u_T - u_{S^{\bar{\imath}}}, \boldsymbol{q}\cdot\boldsymbol{n}_\Gamma)_{S^\Gamma} \Big\}$$

- **WELL-CUT CELLS:** $T \in \mathcal{T}_h^{\mathbf{OK}}$
  - Stencil includes dofs of ill-cut cell(s)

$$\hat{u}_{T^i}^{+} := (\hat{u}_{T^i}, (\hat{u}_{S^i})_{S \in \mathcal{N}_i^{-1}(T)}), \ \forall i \in \{1,2\}$$

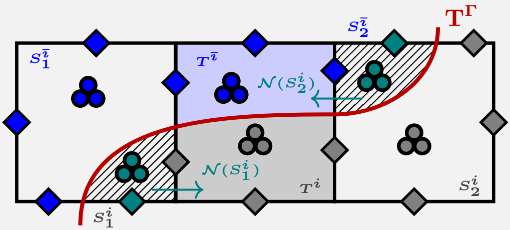

Fig. 10: Pairing configuration

Design of the local gradient reconstruction in the well-cut cells

- **Classical gradient reconstruction:** $\forall i \in \{1,2\}$,

$$(\boldsymbol{G}_{T^i}^k(\hat{u}_T^{+}), \boldsymbol{q})_{T^i} := (\nabla u_{T^i}, \boldsymbol{q})_{T^i} - (u_{T^i} - u_{(\partial T)^i}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{(\partial T)^i} - \delta_{i1}\kappa_1(u_{T^i} - u_{T^{\bar{i}}}, \boldsymbol{q}\cdot\boldsymbol{n}_\Gamma)_{T^\Gamma}$$

  - choice $\delta_{i1}\kappa_1$ robust with respect to strong contrast: $\kappa_1 \ll \kappa_2$

- **Gradient reconstruction with polynomial extension:** $\forall i \in \{1,2\}$,

$$(\boldsymbol{G}_{T^i}^k(\hat{u}_T^{+}), \boldsymbol{q})_{T^i} := (\nabla u_{T^i}, \boldsymbol{q})_{T^i} - (u_{T^i} - u_{(\partial T)^i}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{(\partial T)^i} - \delta_{i1}\kappa_1(u_{T^i} - u_{T^{\bar{i}}}, \boldsymbol{q}\cdot\boldsymbol{n}_\Gamma)_{T^\Gamma}$$

$$+ \sum_{S \in \mathcal{N}_i^{-1}(T)} \left\{ (\nabla u_{T^i}, \boldsymbol{q})_{S^i} - (u_{T^i} - u_{(\partial S)^i}, \boldsymbol{q}\cdot\boldsymbol{n}_S)_{(\partial S)^i} - \delta_{i1}\kappa_1(u_{T^i} - u_{S^{\bar{i}}}, \boldsymbol{q}\cdot\boldsymbol{n}_\Gamma)_{S^\Gamma} \right\}$$

- **ILL-CUT CELLS:** $T \in \mathcal{T}_h^{\mathbf{KO},i}$
  - Stencil of paired cell includes dofs of ill-cut cell(s)

$$\hat{u}^+_{T^{\bar{\imath}}} := (\hat{u}_{T^{\bar{\imath}}}, \hat{u}_{\mathcal{N}(T)^i}, (\hat{u}_{S^{\bar{\imath}}})_{S\in\mathcal{N}_{\bar{\imath}}^{-1}(T)})$$

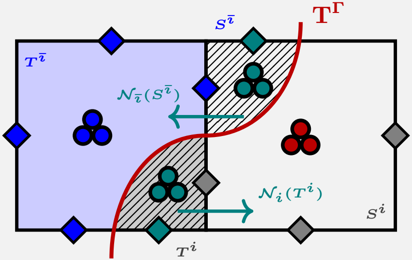

Fig. 11: Pairing configuration

Design of the local gradient reconstruction in the ill-cut cells

- **Classical gradient reconstruction:** $\forall i \in \{1,2\}$,

$$(\boldsymbol{G}^k_{T^i}(\hat{u}^+_T), \boldsymbol{q})_{T^i} := (\nabla u_{T^i}, \boldsymbol{q})_{T^i} - (u_{T^i} - u_{(\partial T)^i}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{(\partial T)^i} - \delta_{i1}\kappa_1(u_{T^i} - u_{T^{\bar{\imath}}}, \boldsymbol{q}\cdot\boldsymbol{n}_\Gamma)_{T^\Gamma}$$

- **Gradient reconstruction with polynomial extension:**

$$(\boldsymbol{G}^k_{T^i}(\hat{u}^+_T), \boldsymbol{q})_{T^i} := 0$$

$$(\boldsymbol{G}^k_{T^{\bar{\imath}}}(\hat{u}^+_T), \overline{\boldsymbol{q}})_{T^{\bar{\imath}}} := (\nabla u_{T^{\bar{\imath}}}, \overline{\boldsymbol{q}})_{T^{\bar{\imath}}} - (u_{T^{\bar{\imath}}} - u_{(\partial T)^{\bar{\imath}}}, \overline{\boldsymbol{q}}\cdot\boldsymbol{n}_T)_{(\partial T)^{\bar{\imath}}} - \delta_{\bar{\imath}1}\kappa_1(u_{T^{\bar{\imath}}} - u_{\mathcal{N}(T)^i}, \overline{\boldsymbol{q}}\cdot\boldsymbol{n}_\Gamma)_{T^\Gamma}$$
$$+ \sum_{S\in\mathcal{N}_{\bar{\imath}}^{-1}(T)} \left\{ (\nabla u_{T^{\bar{\imath}}}, \overline{\boldsymbol{q}})_{S^{\bar{\imath}}} - (u_{T^{\bar{\imath}}} - u_{(\partial S)^{\bar{\imath}}}, \overline{\boldsymbol{q}}\cdot\boldsymbol{n}_S)_{(\partial S)^{\bar{\imath}}} - \delta_{\bar{\imath}1}\kappa_1(u_{T^{\bar{\imath}}} - u_{S^i}, \overline{\boldsymbol{q}}\cdot\boldsymbol{n}_\Gamma)_{S^\Gamma} \right\}$$

HHO stabilization

- **Classical HHO stabilization:** $\forall i \in \{1,2\}$,

$$s_{T^i}(\hat{u}_{T^i},\hat{w}_{T^i}) := \kappa_i h_T^{-1}(\Pi^k_{(\partial T)^i}(u_{T^i}-u_{(\partial T)^i}), w_{T^i}-w_{(\partial T)^i})_{(\partial T)^i}$$

- **Stabilization with polynomial extension (e.g. $T \in \mathcal{T}_h^{\mathbf{OK}}$):** $\forall i \in \{1,2\}$,

$$s_{T^i}(\hat{u}_T^+,\hat{w}_T^+) := \kappa_i h_T^{-1}(\Pi^k_{(\partial T)^i}(u_{T^i}-u_{(\partial T)^i}), w_{T^i}-w_{(\partial T)^i})_{(\partial T)^i} + \sum_{S\in\mathcal{N}_i^{-1}(T)} \kappa_i h_T^{-1}(\Pi^k_{(\partial S)^i}(u_{T^i}-u_{(\partial S)^i}), w_{T^i}-w_{(\partial S)^i})_{(\partial S)^i}$$

Design of the cut stabilization operator (Nitsche's term)

- **Classical cut stabilization operator:** $\forall i \in \{1,2\}$,

$$s^\Gamma_{T^i}(\hat{u}_T^+,\hat{w}_T^+) := \delta_{i1}\kappa_1 h_T^{-1}([\![u_T]\!]_\Gamma, [\![w_T]\!]_\Gamma)_{T^\Gamma}$$

- **Cut stabilization with polynomial extension (e.g. $T \in \mathcal{T}_h^{\mathbf{OK}}$):** $\forall i \in \{1,2\}$,

$$s^\Gamma_{T^i}(\hat{u}_T^+,\hat{w}_T^+) := \delta_{i1}\kappa_1 h_T^{-1}([\![u_T]\!]_\Gamma, [\![w_T]\!]_\Gamma)_{T^\Gamma} + \sum_{S\in\mathcal{N}_i^{-1}(T)} \delta_{i1}\kappa_1 h_T^{-1}([\![u_S]\!]_\Gamma, [\![w_S]\!]_\Gamma)_{S^\Gamma}$$

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Global discrete problem

$$a_h(\hat{u}_h, \hat{w}_h) = \ell_h(\hat{w}_h) \quad \forall \hat{w}_h \in \widehat{\mathcal{U}}_{h0},$$

- $a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} a_{T^i}(\hat{u}_T^+, \hat{w}_T^+)$
  - $a_{T^i}(\hat{u}_T^+, \hat{w}_T^+) := \kappa_i(\boldsymbol{G}^k_{T^i}(\hat{u}_T^+), \boldsymbol{G}^k_{T^i}(\hat{w}_T^+))_{T^i} + s_{T^i}(\hat{u}_T^+, \hat{w}_T^+) + s^\Gamma_{T^i}(\hat{u}_T^+, \hat{w}_T^+)$
- $\ell_h(\hat{w}_h) := \sum_{T \in \mathcal{T}_h^\circ} (f, w_{T^i})_{T^i}$

$$+ \sum_{T \in \mathcal{T}_h^{\mathrm{KO}}} \left\{ (f, w_{\mathcal{N}(T)^i})_{T^i} + (f, w_{T^{\bar{\imath}}})_{T^{\bar{\imath}}} \right\} + \sum_{T \in \mathcal{T}_h^{\mathrm{OK}}} \sum_{i \in \{1,2\}} (f, w_{T^i})_{T^i}$$

  - For simplicity, we consider $g_D = g_N = 0$

Algebraic realization for gradient reconstruction

- Algebraic realization of $(\boldsymbol{G}^k_{T^i}(\hat{u}^+_T), \boldsymbol{G}^k_{T^i}(\hat{w}^+_T))_{T^i}$ (e.g. $\forall T \in \mathcal{T}_h^{\mathrm{OK}}$):

$$\forall i \in \{1,2\}, \qquad \mathbb{G}^\dagger_{T^i}\mathbf{M}^{-1}_{T^i}\mathbb{G}_{T^i} := \mathbf{G}^\dagger_{T^i}\mathbf{M}^{-1}_{T^i}\mathbf{G}_{T^i} + \sum_{S\in\mathcal{N}_i^{-1}(T)} \{\mathbf{G}^\dagger_{S^i}\mathbf{M}^{-1}_{T^i}\mathbf{G}_{S^i}\}$$

  - $\mathbf{M}_T := (\phi_{T,i}, \phi_{T,j})_T, \;\; 0 \le i,j < N^k := \dim(\mathbb{P}^k(T;\mathbb{R})),$ (componentwise mass matrix)
    - $N^k_d := d \times N^k$
    - $N_S := \#\mathcal{N}_i^{-1}(T)$
    - $N_{\partial T} :=$ number of faces of T
    - $N_{\partial S} :=$ number of faces of S
  - $\mathbb{G}_T :=$ matrix with N^k_d rows and column blocks of widths $N^{k'}_d$, $N_{\partial T} \times N^k_{d-1}$, $N_S \times N^{k'}_d$, $N_S \times N_{\partial S} \times N^{k'}_d$
  - **Extension of local bilinear form $\longrightarrow$ Modification of assembly**

Error analysis

- Based on [Burman, Cicuttin, Delay, and Ern, 2021]
  - Stability (coercivity)
  - Consistency
  - Quasi-optimal error estimates
  - For smooth solution, H^1-error: $\mathcal{O}(h^{k+1})$

- Implementation in progress

Thank you for your attention !