Hybrid high-order methods for the numerical simulation of elasto-acoustic wave propagation

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Goal

- Accurate modeling and simulation of seismo-acoustic waves through **heterogeneous domains with complex geometries**

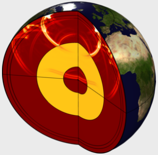


Fig. 1: Global seismic wave propagation

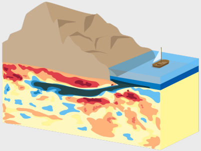


Fig. 2: Local heterogeneities of the Earth

- Minimize numerical dispersion and dissipation for long time propagation

Commonly used numerical tools

- Spectral Element Method (cG) / Finite Differences (FDTD)
- **Main issue:** Complex mesh generation for realistic geological structures

cG vs. dG methods

Main advantages of dG methods

- **Mesh flexibility:** Handling of unstructured / polyhedral meshes
- **Local conservativity at the element level**
- **Same order of convergence as cG for smooth solutions:**
  - H^1-error: $\mathcal{O}(h^k)$
  - L^2-error: $\mathcal{O}(h^{k+1})$

Drawbacks of dG methods

- **Higher computational cost and memory requirement**

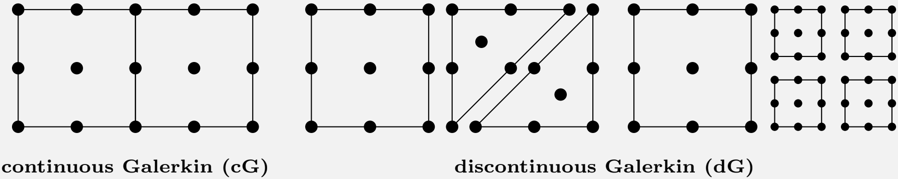


Fig. 3: Discrete unknowns for cG and dG methods

Introduction to HHO methods

- **Seminal papers:** [Di Pietro, Ern, and Lemaire, 2014], [Di Pietro and Ern, 2015]

Degrees of freedom

- **Polynomial unknowns attached to mesh cells and faces**

HHO unknowns:

$$\hat{u}_h := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \hat{\mathcal{U}}_h$$

● **Cell unknowns, degree $k' \in \{k, k+1\}$** ◆ **Face unknowns, degree $k \geq 0$**

Fig. 4: Local HHO unknowns. **Left:** $k' = k = 0$. **Right:** $k' = k + 1 = 1$.

- Equal-order: $k' = k$
- Mixed-order: $k' = k + 1$

Design

- **Gradient reconstruction operator:**

  $$(\boldsymbol{\nabla u})_{|T} \to \mathbf{G}_{\boldsymbol{T}}(\hat{\boldsymbol{u}}_T) \in \mathbb{P}^k(T;\mathbb{R}^d)$$

  Design of $\mathbf{G}_{\boldsymbol{T}}(\hat{\boldsymbol{u}}_T)$ mimics an integration by parts

- **Stabilization operator:** $\boldsymbol{\delta}_{\partial T}(\hat{\boldsymbol{u}}_T) := \boldsymbol{u}_{\partial T} - \boldsymbol{u}_{T|\partial T} \approx \mathbf{0}$

  Matching of cell dofs trace with face dofs (weakly)

Advantages of HHO over dG methods

- **Improved error estimates for smooth solutions**
  - H^1-error: $\mathcal{O}\big(h^{k+1}\big)$
  - L^2-error: $\mathcal{O}\big(h^{k+2}\big)$

- **Attractive computational costs**

  Elimination of cell unknowns by **static condensation**
  - Global problem couples only face dofs
  - Cell dofs recovered by local post-processing

Link to other methods

$$\text{HHO} \equiv \text{HDG} \equiv \text{WG} \equiv \text{ncVEM}$$

[Cockburn, Di Pietro, and Ern, 2016] [Lemaire, 2020] [Cicuttin, Ern, and Pignet, 2021]

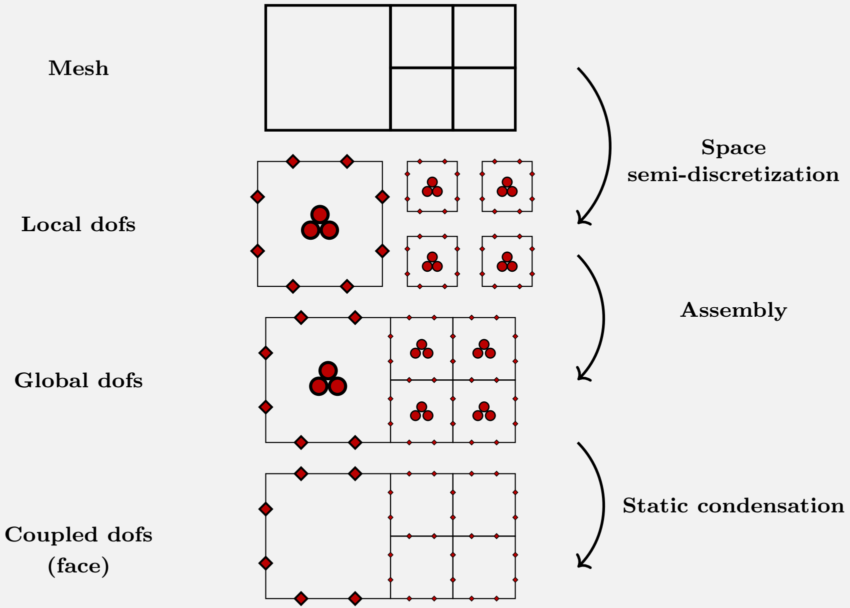
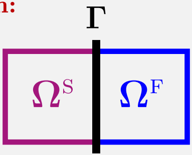


Fig. 5: Assembly and static condensation procedure in HHO framework

Model problem:



$\Omega := \Omega^{\mathrm{S}} \cup \Omega^{\mathrm{F}}$

Elasto-acoustic interface Γ

Fig. 6: Setting for elasto-acoustic coupling

Strong form of acoustic and elastic wave equation in 1^{st} order formulation

$$\begin{cases} \partial_t \boldsymbol{\varepsilon} - \nabla_s \boldsymbol{v}^{\mathrm{S}} = \mathbf{0} \\ \rho^{\mathrm{S}} \partial_t \boldsymbol{v}^{\mathrm{S}} - \nabla \cdot (\boldsymbol{\mathcal{C}}{:}\boldsymbol{\varepsilon}) = \boldsymbol{f}^{\mathrm{S}} \end{cases}$$

$$\begin{cases} \rho^{\mathrm{F}} \partial_t \boldsymbol{v}^{\mathrm{F}} - \nabla p = \mathbf{0} \\ \dfrac{1}{\kappa} \partial_t p - \nabla \cdot \boldsymbol{v}^{\mathrm{F}} = f^{\mathrm{F}} \end{cases}$$

Unknowns

- $\boldsymbol{v}^{\mathrm{S}}$ elastic velocity field
- $\boldsymbol{\varepsilon} := \nabla_s \boldsymbol{u}$ linearized strain tensor
- p scalar pressure field
- $\boldsymbol{v}^{\mathrm{F}}$ acoustic velocity field

Parameters

- ρ^{S}, $\boldsymbol{\mathcal{C}}(\lambda, \mu)$ (Lamé coefficients)
- $c_{\mathrm{P}}^{\mathrm{S}} := \sqrt{\dfrac{\lambda + 2\mu}{\rho^{\mathrm{S}}}}$, $c_{\mathrm{S}} := \sqrt{\dfrac{\mu}{\rho^{\mathrm{S}}}}$
- ρ^{F}, κ
- $c_{\mathrm{P}}^{\mathrm{F}} := \sqrt{\dfrac{\kappa}{\rho^{\mathrm{F}}}}$

Coupling conditions

$$\begin{cases} \boldsymbol{v}^{\mathrm{S}} \cdot \boldsymbol{n}_\Gamma = \boldsymbol{v}^{\mathrm{F}} \cdot \boldsymbol{n}_\Gamma \\ (\boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}) \cdot \boldsymbol{n}_\Gamma = p\, \boldsymbol{n}_\Gamma \end{cases}$$

- ▶ Balance of mass + Non-penetration condition
- ▶ Balance of forces

Initial and boundary conditions

- **Initial conditions on** $(\rho^{\mathrm{S}}, \boldsymbol{v}^{\mathrm{S}})$ **and** $(\rho^{\mathrm{F}}, \boldsymbol{v}^{\mathrm{F}})$
- **Homogeneous Dirichlet boundary conditions on $\partial\Omega$ for simplicity**

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HHO space semi-discretization

- **Elastic domain:** $\underbrace{\boldsymbol{\mathcal{Z}}^{k'}_{\mathcal{T}^{\mathrm{S}}} := \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})}_{\text{space for } \boldsymbol{\varepsilon}}$, $\quad \underbrace{\widehat{\boldsymbol{\mathcal{U}}}^{\mathrm{S}}_h := \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^{k'}(T;\mathbb{R}^d) \times \bigtimes_{F\in\mathcal{F}^{\mathrm{S}}_h} \mathbb{P}^k(F;\mathbb{R}^d)}_{\text{space for } \boldsymbol{v}^{\mathrm{S}}}$
- **Acoustic domain:** $\underbrace{\boldsymbol{\mathcal{V}}^{k}_{\mathcal{T}^{\mathrm{F}}} := \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^k(T;\mathbb{R}^{d})}_{\text{space for } \boldsymbol{v}^{\mathrm{F}}}$, $\quad \underbrace{\widehat{\mathcal{U}}^{\mathrm{F}}_h := \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}^{k'}(T;\mathbb{R}) \times \bigtimes_{F\in\mathcal{F}^{\mathrm{F}}_h} \mathbb{P}^k(F;\mathbb{R})}_{\text{space for } p}$

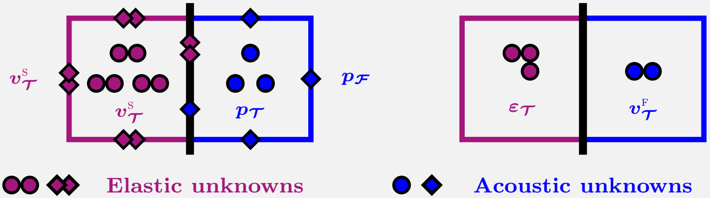

Fig. 7: Elasto-acoustic unknowns with $k'=1$ and $k=0$. **Left:** HHO unknowns for $\boldsymbol{v}^{\mathrm{S}}$ and p. **Right:** dG unknowns for $\boldsymbol{\varepsilon}$ and $\boldsymbol{v}^{\mathrm{F}}$.

References

- Same discretization as for acoustic [Burman, Duran, and Ern, 2022] and elastic [Burman, Duran, Ern, and Steins, 2021] problems, but **with coupling terms**

Local reconstruction operators

- **Strain reconstruction:** $\boldsymbol{E}_T(\hat{\boldsymbol{v}}_T^{\mathrm{S}}) \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ s.t. for all $\hat{\boldsymbol{v}}_T^{\mathrm{S}} \in \widehat{\boldsymbol{\mathcal{U}}}_T^{\mathrm{S}}$,

$$(\boldsymbol{E}_T(\hat{\boldsymbol{v}}_T^{\mathrm{S}}),\boldsymbol{\zeta})_T = (\nabla_s \boldsymbol{v}_T^{\mathrm{S}},\boldsymbol{\zeta})_T - (\boldsymbol{v}_T^{\mathrm{S}} - \boldsymbol{v}_{\partial T}^{\mathrm{S}}, \boldsymbol{\zeta}\cdot\boldsymbol{n}_T)_{\partial T}, \quad \forall \boldsymbol{\zeta} \in \mathbb{P}^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

- **Gradient reconstruction:** $\boldsymbol{G}_T(\hat{p}_T) \in \mathbb{P}^k(T;\mathbb{R}^d)$ s.t. for all $\hat{p}_T \in \widehat{\mathcal{U}}_T^{\mathrm{F}}$,

$$(\boldsymbol{G}_T(\hat{p}_T),\boldsymbol{q})_T = (\nabla p_T,\boldsymbol{q})_T - (p_T - p_{\partial T}, \boldsymbol{q}\cdot\boldsymbol{n}_T)_{\partial T}, \quad \forall \boldsymbol{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$$

Local stabilization operators

- **Mixed-order discretization: Stabilization in HDG** (Lehrenfeld-Schöberl)

$$S_{\partial T}(\hat{p}_T) := \Pi^k_{\partial T}(p_T - p_{\partial T}) \qquad \boldsymbol{S}_{\partial T}(\hat{\boldsymbol{v}}_T^{\mathrm{S}}) := \boldsymbol{\Pi}^k_{\partial T}(\boldsymbol{v}_T^{\mathrm{S}} - \boldsymbol{v}_{\partial T}^{\mathrm{S}})$$

- **Equal-order discretization: Specific stabilization to HHO**
  - Needs additional velocity and pressure reconstructions

HHO space semi-discretization for the elasto-acoustic coupling

- **Elastic wave equation:** $\boldsymbol{E}_{\mathcal{T}}(\hat{\boldsymbol{v}}_h^{\mathrm{S}})_{|T} := \boldsymbol{E}_T(\hat{\boldsymbol{v}}_T^{\mathrm{S}})$

$$(\partial_t \boldsymbol{\varepsilon}_{\mathcal{T}}(t), \boldsymbol{z}_{\mathcal{T}})_{\Omega^{\mathrm{S}}} - (\boldsymbol{E}_{\mathcal{T}}(\hat{\boldsymbol{v}}_h^{\mathrm{S}}(t)), \boldsymbol{z}_{\mathcal{T}})_{\Omega^{\mathrm{S}}} = 0$$

$$(\rho^{\mathrm{S}} \partial_t \boldsymbol{v}_{\mathcal{T}^{\mathrm{S}}}^{\mathrm{S}}(t), \boldsymbol{w}_{\mathcal{T}})_{\Omega^{\mathrm{S}}} + (\boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}_{\mathcal{T}}, \boldsymbol{E}_{\mathcal{T}}(\hat{\boldsymbol{w}}_h))_{\Omega^{\mathrm{S}}} + s_h^{\mathrm{S}}(\hat{\boldsymbol{v}}_h^{\mathrm{S}}, \hat{\boldsymbol{w}}_h) + (p_{\mathcal{F}}(t), \boldsymbol{w}_{\mathcal{F}} \cdot \boldsymbol{n}_{\Gamma})_{\Gamma} = (\boldsymbol{f}^{\mathrm{S}}(t), \boldsymbol{w}_{\mathcal{T}})_{\Omega^{\mathrm{S}}}$$

- **Acoustic wave equation:** $\boldsymbol{G}_{\mathcal{T}}(\hat{p}_h)_{|T} := \boldsymbol{G}_T(\hat{p}_T)$

$$(\rho^{\mathrm{F}} \partial_t \boldsymbol{v}^{\mathrm{F}} t(t), \boldsymbol{r}_{\mathcal{T}})_{\Omega^{\mathrm{F}}} + (\boldsymbol{G}_{\mathcal{T}}(\hat{p}_h(t)), \boldsymbol{r}_{\mathcal{T}})_{\Omega^{\mathrm{F}}} = 0$$

$$(\tfrac{1}{\kappa} \partial_t p_{\mathcal{T}}(t), q_{\mathcal{T}})_{\Omega^{\mathrm{F}}} - (\boldsymbol{v}^{\mathrm{F}} t(t), \boldsymbol{G}_{\mathcal{T}}(\hat{q}_h))_{\Omega^{\mathrm{F}}} + s_h^{\mathrm{F}}(\hat{p}_h(t), \hat{q}_h) - (\boldsymbol{v}^{\mathrm{F}} s(t) \cdot \boldsymbol{n}_{\Gamma}, q_{\mathcal{F}})_{\Gamma} = (f^{\mathrm{F}}(t), q_{\mathcal{T}})_{\Omega^{\mathrm{F}}}$$

Global stabilization forms

$$s_h^{\mathrm{S}}(\hat{\boldsymbol{v}}_h^{\mathrm{S}}, \hat{\boldsymbol{\zeta}}_h) = \sum_{T \in \mathcal{T}_h} \tau_T^{\mathrm{S}} (\boldsymbol{S}_{\partial T}(\hat{\boldsymbol{v}}_T^{\mathrm{S}}), \boldsymbol{S}_{\partial T}(\hat{\boldsymbol{\zeta}}_T))_{\partial T}$$

$$s_h^{\mathrm{F}}(\hat{p}_h, \hat{q}_h) = \sum_{T \in \mathcal{T}_h} \tau_T^{\mathrm{F}} (\boldsymbol{S}_{\partial T}(\hat{p}_T), \boldsymbol{S}_{\partial T}(\hat{\boldsymbol{q}}_T))_{\partial T}$$

- **with two strategies:** $\tau_T^{\mathrm{S}} = \mathcal{O}(1) = \tau_T^{\mathrm{F}}$ or $\tau_T^{\mathrm{S}} = \mathcal{O}(1/h) = \tau_T^{\mathrm{F}}$

Algebraic realization

- Static coupling between cell and face unknowns

$$
\left[\begin{array}{ccc|ccc}
\mathbf{M}^{v^{\mathrm{F}}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 & 0\\
0 & \mathbf{M}^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & \mathbf{M}^{\varepsilon}_{\mathcal{T}\mathcal{T}} & 0 & 0\\
0 & 0 & 0 & 0 & \mathbf{M}^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\frac{\mathrm{d}}{\mathrm{dt}}
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F}}_{\mathcal{T}^{\mathrm{F}}}\\
\mathrm{P}_{\mathcal{T}^{\mathrm{F}}}\\
\mathrm{P}_{\mathcal{F}^{\mathrm{F}}}\\
\hline
\mathrm{S}_{\mathcal{T}^{\mathrm{S}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{T}^{\mathrm{S}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right]
+
\left[\begin{array}{ccc|ccc}
0 & -\mathrm{G}_{\mathcal{T}} & -\mathrm{G}_{\mathcal{F}} & 0 & 0 & 0\\
\mathrm{G}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{F}} & 0 & 0 & 0\\
\mathrm{G}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{T}} & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & 0 & 0 & \mathrm{C}_{\Gamma}\\
\hline
0 & 0 & 0 & 0 & -\mathrm{E}_{\mathcal{T}} & -\mathrm{E}_{\mathcal{F}}\\
0 & 0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{F}}\\
0 & 0 & -\mathrm{C}^{\dagger}_{\Gamma} & \mathrm{E}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{T}} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}}
\end{array}\right]
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F}}_{\mathcal{T}^{\mathrm{F}}}\\
\mathrm{P}_{\mathcal{T}^{\mathrm{F}}}\\
\mathrm{P}_{\mathcal{F}^{\mathrm{F}}}\\
\hline
\mathrm{S}_{\mathcal{T}^{\mathrm{S}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{T}^{\mathrm{S}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right]
=
\left[\begin{array}{c}
0\\
\mathrm{F}^{\mathrm{F}}_{\mathcal{T}^{\mathrm{F}}}\\
0\\
\hline
0\\
\mathrm{F}^{\mathrm{S}}_{\mathcal{T}^{\mathrm{S}}}\\
0
\end{array}\right]
$$

- Rearrangement of dofs: cell unknowns first and then face unknowns

$$
\left[\begin{array}{cc|cc|cc}
\mathbf{M}^{v^{\mathrm{F}}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 & 0\\
0 & \mathbf{M}^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0\\
\hline
0 & 0 & \mathbf{M}^{\varepsilon}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0\\
0 & 0 & 0 & \mathbf{M}^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\frac{\mathrm{d}}{\mathrm{dt}}
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F}}_{\mathcal{T}^{\mathrm{F}}}\\
\mathrm{P}_{\mathcal{T}^{\mathrm{F}}}\\
\hline
\mathrm{S}_{\mathcal{T}^{\mathrm{S}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{T}^{\mathrm{S}}}\\
\hline
\mathrm{P}_{\mathcal{F}^{\mathrm{F}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right]
+
\left[\begin{array}{cc|cc|cc}
0 & -\mathrm{G}_{\mathcal{T}} & 0 & 0 & -\mathrm{G}_{\mathcal{F}} & 0\\
\mathrm{G}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{F}} & 0\\
\hline
0 & 0 & 0 & -\mathrm{E}_{\mathcal{T}} & 0 & -\mathrm{E}_{\mathcal{F}}\\
0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{F}}\\
\hline
\mathrm{G}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{T}} & 0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & \mathrm{C}_{\Gamma}\\
0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{T}} & -\mathrm{C}^{\dagger}_{\Gamma} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}}
\end{array}\right]
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F}}_{\mathcal{T}^{\mathrm{F}}}\\
\mathrm{P}_{\mathcal{T}^{\mathrm{F}}}\\
\hline
\mathrm{S}_{\mathcal{T}^{\mathrm{S}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{T}^{\mathrm{S}}}\\
\hline
\mathrm{P}_{\mathcal{F}^{\mathrm{F}}}\\
\mathrm{V}^{\mathrm{S}}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right]
=
\left[\begin{array}{c}
0\\
\mathrm{F}^{\mathrm{F}}_{\mathcal{T}^{\mathrm{F}}}\\
\hline
0\\
\mathrm{F}^{\mathrm{S}}_{\mathcal{T}^{\mathrm{S}}}\\
\hline
0\\
0
\end{array}\right]
$$

SDIRK(s, s + 1) schemes

- Generic ODE with $f : J \times \mathbb{R}^m \to \mathbb{R}^m$,

$$\begin{cases} y'(t) = f(t, y(t)), & \forall t \in J := [0, T) \\ y_{|t=0} = y_0 \in \mathbb{R}^m \end{cases}$$

- SDIRK$(s, s+1)$ consists
  - in solving sequentially for all $i \in \{1, ..., s\}$,

$$u_{\boldsymbol{i}}^{[n]} = u_{n-1} + \Delta t \sum_{j=1}^{\boldsymbol{i}} a_{ij} f(t_{n-1} + c_j \Delta t,\ u_j^{[n]})$$

  - and setting

$$u_n := u_{n-1} + \Delta t \sum_{j=1}^{s} b_j f(t_{n-1} + c_j \Delta t,\ u_j^{[n]})$$

$$\begin{array}{c|cccc} c_1 & a_* & 0 & \cdots & 0 \\ c_2 & a_{21} & a_* & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{s,s-1} & a_* \\ \hline & b_1 & \cdots & b_{s-1} & b_s \end{array}$$

Algebraic realization of SDIRK-HHO

- Face-based sparse linear system to be solved at each stage
- We solve sequentially for all $i \in \{1, \dots, s\}$,

$$
\left[\begin{array}{cc:cc:cc}
\mathbf{M}^{v^{\mathrm{F}}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{M}^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & \mathbf{M}^{\varepsilon}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{M}^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F},n,i}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{P}^{n,i}_{\mathcal{T}^{\mathrm{F}}} \\ \hdashline
\mathrm{S}^{n,i}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{V}^{\mathrm{S},n,i}_{\mathcal{T}^{\mathrm{S}}} \\ \hdashline
\mathrm{P}^{n,i}_{\mathcal{F}^{\mathrm{F}}} \\
\mathrm{V}^{\mathrm{S},n,i}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right]
=
\left[\begin{array}{cc:cc:cc}
\mathbf{M}^{v^{\mathrm{F}}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{M}^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & \mathbf{M}^{\varepsilon}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{M}^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F},n-1}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{P}^{n-1}_{\mathcal{T}^{\mathrm{F}}} \\ \hdashline
\mathrm{S}^{n-1}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{V}^{\mathrm{S},n-1}_{\mathcal{T}^{\mathrm{S}}} \\ \hdashline
\mathrm{P}^{n-1}_{\mathcal{F}^{\mathrm{F}}} \\
\mathrm{V}^{\mathrm{S},n-1}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right]
$$

$$
+\Delta t \sum_{j=1}^{i} a_{ij} \left(\left[\begin{array}{c}
0 \\
\mathrm{F}^{\mathrm{F},n-1+c_j}_{\mathcal{T}^{\mathrm{F}}} \\ \hdashline
0 \\
\mathrm{F}^{\mathrm{S},n-1+c_j}_{\mathcal{T}^{\mathrm{F}}} \\ \hdashline
0 \\
0
\end{array}\right]
-
\left[\begin{array}{cc:cc:cc}
0 & -\mathrm{G}_{\mathcal{T}} & 0 & 0 & -\mathrm{G}_{\mathcal{F}} & 0 \\
\mathrm{G}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{F}} & 0 \\ \hdashline
0 & 0 & 0 & -\mathrm{E}_{\mathcal{T}} & 0 & -\mathrm{E}_{\mathcal{F}} \\
0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{F}} \\ \hdashline
\mathrm{G}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{T}} & 0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & \mathrm{C}^{\Gamma} \\
0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{T}} & -\mathrm{C}^{\dagger}_{\Gamma} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}}
\end{array}\right]
\left[\begin{array}{c}
\mathrm{V}^{\mathrm{F},n,j}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{P}^{n,j}_{\mathcal{T}^{\mathrm{F}}} \\ \hdashline
\mathrm{S}^{n,j}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{V}^{\mathrm{S},n,j}_{\mathcal{T}^{\mathrm{S}}} \\ \hdashline
\mathrm{P}^{n,j}_{\mathcal{F}^{\mathrm{F}}} \\
\mathrm{V}^{\mathrm{S},n,j}_{\mathcal{F}^{\mathrm{S}}}
\end{array}\right] \right)
$$

- The upper 4×4 submatrix associated with all the cell unknowns is block-diagonal
  - Schur complement procedure

ERK(s) schemes

- ERK(s) consists
  - in updating sequentially for all $i \in \{1, ..., s\}$,

$$u_i^{[n]} = u_{n-1} + \Delta t \sum_{j=1}^{\boldsymbol{i-1}} a_{ij} f(t_{n-1} + c_j \Delta t,\ u_j^{[n]})$$

  - and setting

$$u_n := u_{n-1} + \Delta t \sum_{j=1}^{s} b_j f\left(t_{n-1} + c_j \Delta t,\ u_j^{[n]}\right)$$

$$\begin{array}{c|cccc} c_1 & 0 & \cdots & \cdots & 0 \\ c_2 & a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & \cdots & b_{s-1} & b_s \end{array}$$

HHO-ERK scheme

- Coupling of face unknowns at the interface Γ

$$
\begin{bmatrix}
\mathbf{M}^{v^{\mathrm{F}}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{M}^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{M}^{\varepsilon}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{M}^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathrm{V}^{\mathrm{F},n,i}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{P}^{n,i}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{S}^{n,i}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{V}^{\mathrm{S},n,i}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{P}^{n,i}_{\mathcal{F}^{\mathrm{F}}} \\
\mathrm{V}^{\mathrm{S},n,i}_{\mathcal{F}^{\mathrm{S}}}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{M}^{v^{\mathrm{F}}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{M}^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{M}^{\varepsilon}_{\mathcal{T}\mathcal{T}} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{M}^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathrm{V}^{\mathrm{F},n-1}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{P}^{n-1}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{S}^{n-1}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{V}^{\mathrm{S},n-1}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{P}^{n-1}_{\mathcal{F}^{\mathrm{F}}} \\
\mathrm{V}^{\mathrm{S},n-1}_{\mathcal{F}^{\mathrm{S}}}
\end{bmatrix}
$$

$$
+\Delta t \sum_{j=1}^{i-1} a_{ij}
\left(
\begin{bmatrix}
0 \\
\mathrm{F}^{\mathrm{F},n-1+c_j}_{\mathcal{T}^{\mathrm{F}}} \\
0 \\
\mathrm{F}^{\mathrm{S},n-1+c_j}_{\mathcal{T}^{\mathrm{S}}} \\
0 \\
0
\end{bmatrix}
-
\begin{bmatrix}
0 & -\mathrm{G}_{\mathcal{T}} & 0 & 0 & -\mathrm{G}_{\mathcal{F}} & 0 \\
\mathrm{G}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{T}} & 0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{T}\mathcal{F}} & 0 \\
0 & 0 & 0 & -\mathrm{E}_{\mathcal{T}} & 0 & -\mathrm{E}_{\mathcal{F}} \\
0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{T}} & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{T}} & 0 & \Sigma^{\mathrm{S}}_{\mathcal{T}\mathcal{F}} \\
\mathrm{G}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{T}} & 0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & \mathrm{C}^{\Gamma} \\
0 & 0 & \mathrm{E}^{\dagger}_{\mathcal{F}} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{T}} & -\mathrm{C}^{\dagger}_{\Gamma} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}}
\end{bmatrix}
\begin{bmatrix}
\mathrm{V}^{\mathrm{F},n,j}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{P}^{n,j}_{\mathcal{T}^{\mathrm{F}}} \\
\mathrm{S}^{n,j}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{V}^{\mathrm{S},n,j}_{\mathcal{T}^{\mathrm{S}}} \\
\mathrm{P}^{n,j}_{\mathcal{F}^{\mathrm{F}}} \\
\mathrm{V}^{\mathrm{S},n,j}_{\mathcal{F}^{\mathrm{S}}}
\end{bmatrix}
\right)
$$

- **Key observation:**

$$
\begin{bmatrix}
\Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & \mathrm{C}^{\Gamma} \\
-\mathrm{C}^{\dagger}_{\Gamma} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}}
\end{bmatrix}
$$
has a block-diagonal structure for **mixed-order** HHO

Rearrangement of the face terms for the inversion of coupling block

- Distinguish between internal faces in $\Omega^{\mathrm{S}} \cup \Omega^{\mathrm{F}}$ and faces located on Γ

$$
\left[\begin{array}{cc:cc}
\Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & 0 & 0 & 0 \\
0 & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}} & 0 & 0 \\
\hdashline
0 & 0 & \Sigma^{\mathrm{F}}_{\mathcal{F}\mathcal{F}} & \mathrm{C}_{\Gamma} \\
0 & 0 & -\mathrm{C}^{\dagger}_{\Gamma} & \Sigma^{\mathrm{S}}_{\mathcal{F}\mathcal{F}}
\end{array}\right]
\left[\begin{array}{c}
\mathrm{P}_{\mathcal{F}_h^{\circ \mathrm{F}}} \\
\mathrm{V}^{\mathrm{S}}_{\mathcal{F}_h^{\circ \mathrm{S}}} \\
\hdashline
\mathrm{P}_{\mathcal{F}_h^{\circ \Gamma}} \\
\mathrm{V}^{\mathrm{S}}_{\mathcal{F}_h^{\circ \Gamma}}
\end{array}\right]
$$

$$
\left[\begin{array}{cc:cc:cc}
\Sigma^{\mathrm{F}}_{F^1} & \mathrm{C}_{F^1} & 0 & 0 & 0 & 0 \\
-\mathrm{C}^{\dagger}_{F^1} & \Sigma^{\mathrm{S}}_{F^1} & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & \ddots & & 0 & 0 \\
0 & 0 & & \ddots & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & \Sigma^{\mathrm{F}}_{F^n} & \mathrm{C}_{F^n} \\
0 & 0 & 0 & 0 & -\mathrm{C}^{\dagger}_{F^n} & \Sigma^{\mathrm{S}}_{F^n}
\end{array}\right]
\left[\begin{array}{c}
\mathrm{P}_{F^1} \\
\mathrm{V}^{\mathrm{S}}_{F^1} \\
\hdashline
\vdots \\
\vdots \\
\hdashline
\mathrm{P}_{F^n} \\
\mathrm{V}^{\mathrm{S}}_{F^n}
\end{array}\right]
$$

Table of Contents

Computational parameters

- Space refinement: $h = 0.1 \times 2^{-\ell}$ (in each subdomain)
- Time refinement: $\Delta t = 0.1 \times 2^{-n}$

Meshes

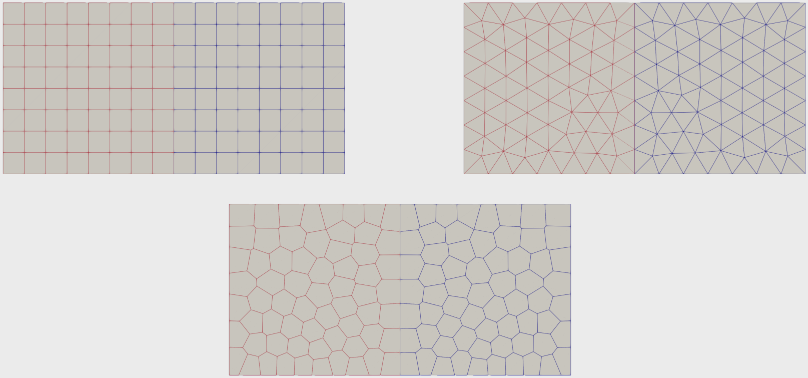

Fig. 8: Cartesian, simplicial and polyhedral meshes for $\ell = 0$

Convergence rates in time

- **Analytical solution:** polynomial in space, sinusoidal in time

- **SDIRK-HHO scheme**
  - $k' = k + 1 = 6$
  - $\ell = 2$
  - $n \in \{3, 4, 5, 6, 7\}$
  - $\tau^{\mathrm{F}} = \mathcal{O}(1) = \tau^{\mathrm{S}}$

- **ERK-HHO scheme**
  - $k' = k + 1 = 5$
  - $\ell = 1$
  - $n \in \{6, 7, 8, 9\}$
  - $\tau^{\mathrm{F}} = \mathcal{O}(1) = \tau^{\mathrm{S}}$

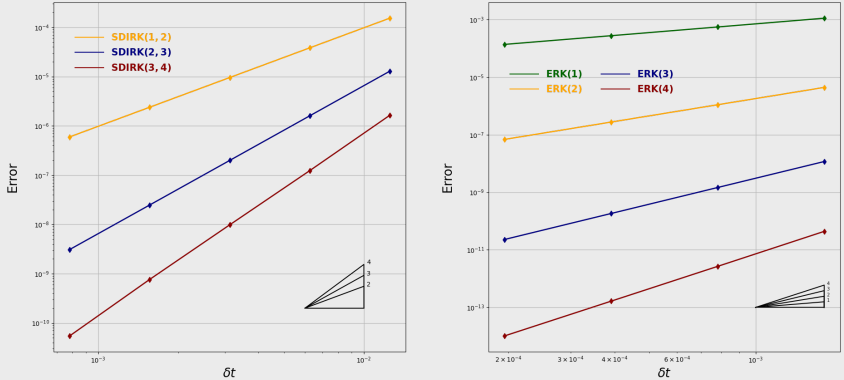

Fig. 9: $\boldsymbol{L}^2$-errors for HHO-RK schemes as a function of the time-step

Convergence rates in space

- **Analytical solution:** polynomial in time, sinusoidal in space
- **SDIRK(3,4)-HHO scheme**
- $n = 8$
- $\ell \in \{0, 1, 2, 3, 4\}$

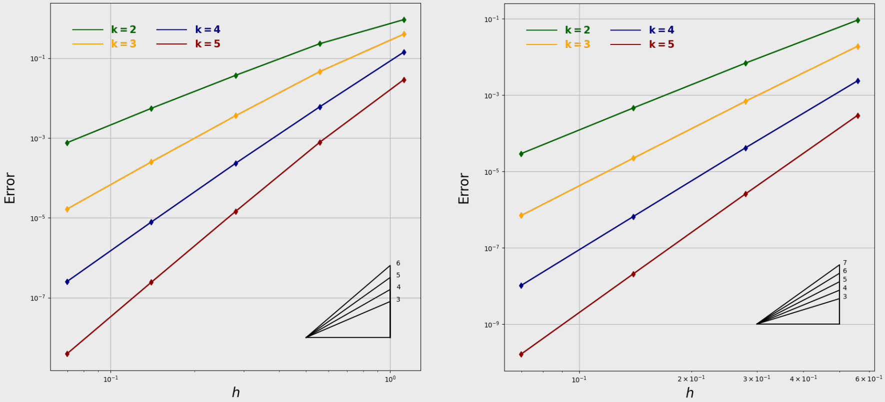

Fig. 10: $\boldsymbol{L}^2$-errors for the HHO-SDIRK(3,4) schemes as a function of the mesh-size. **Left:** $\tau_T^{\mathrm{F}} = \mathcal{O}(1) = \tau_T^{\mathrm{S}}$. **Right:** $\tau_T^{\mathrm{F}} = \mathcal{O}(h_T^{-1}) = \tau_T^{\mathrm{S}}$

Ricker wavelet

- **SDIRK(3,4)**, $k = 1$, $\ell = 7$, $n = 9$
- Homogeneous Dirichlet boundary conditions
- **Initial condition:** velocity Ricker wavelet centered at point $(x_c, y_c) \in \Omega_s cf$,

$$\boldsymbol{v_0}(x, y) := \theta e^{-\pi^2 \frac{r^2}{\lambda^2}} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

Academic test case

- **Homogeneous material properties:** $\rho^{\mathrm{F}} = \rho^{\mathrm{S}} = 1$, $c_{\mathrm{P}}^{\mathrm{S}} = \sqrt{3}$, $c_{\mathrm{P}}^{\mathrm{F}} = c_{\mathrm{S}} = 1$

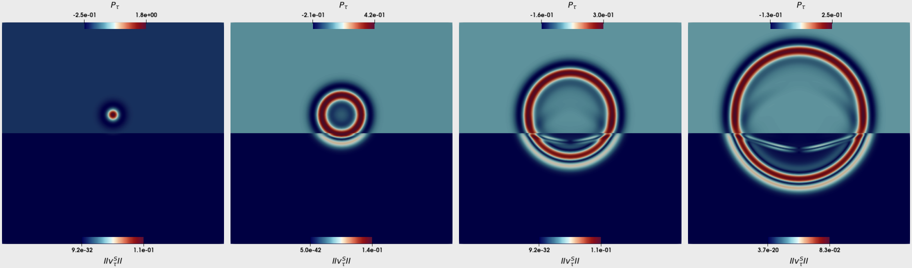

Fig. 11: Acoustic pressure (upper side) and elastic velocity norm (lower side) at times $t \in \{0, 0.025, 0.075, 0.15\}$

Realistic test case with strong property contrast: Granite-Water

Material properties:

- **Granite:** $\rho^{\mathrm{S}} = 2800$ kg.m^{-3}, $c_{\mathrm{P}}^{\mathrm{S}} = 5000$ m.s^{-1}, $c_{\mathrm{S}} = 3000$ m.s^{-1}
- **Water:** $\rho^{\mathrm{F}} = 997$ kg.m^{-3}, $\kappa = 2.1 \times 10^9$ Pa, $c_{\mathrm{P}}^{\mathrm{F}} = 1450$ m.s^{-1}

Computational parameters: SDIRK(3,4), $n = 8$, $l = 7$, $k = 2$

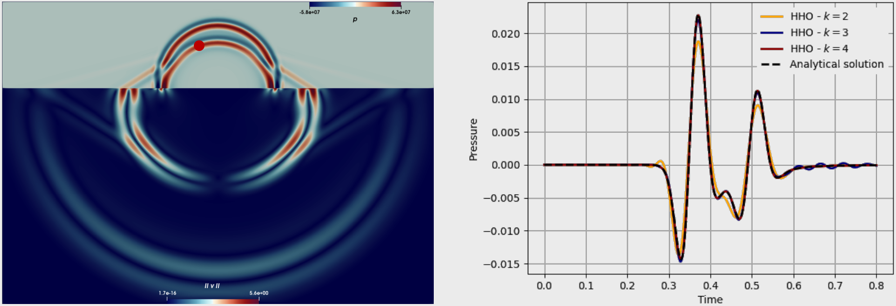

Fig. 12: **Left panel:** Acoustic pressure (upper side) and elastic velocity norm (lower side) at time $t = 0.375$s. **Right panel:** Comparison to analytical solution (Gar6more).

Propagation of an elastic pulse in sedimentary basin and atmosphere

- **Material properties:**
  - **Sedimentary basin:** $\rho^{\mathrm{S}} = 1200$ kg.m^{-3}, $c_{\mathrm{P}}^{\mathrm{S}} = 3400$ m.s^{-1}, $c_{\mathrm{S}} = 1400$ m.s^{-1}
  - **Bedrock:** $\rho^{\mathrm{S}} = 5350$ kg.m^{-3}, $c_{\mathrm{P}}^{\mathrm{S}} = 3090$ m.s^{-1}, $c_{\mathrm{S}} = 2570$ m.s^{-1}
  - **Air:** $\rho^{\mathrm{F}} = 1.292$ kg.m^{-3}, $c_{\mathrm{P}}^{\mathrm{F}} = 340$ m.s^{-1}
- **Computational parameters:** SDIRK(3,4), $k = 1$, $\ell = 8$, $n = 9$
- **Homogeneous Dirichlet boundary conditions**
- **Initial condition:** velocity Ricker wavelet centered at point $(x_c, y_c) \in \Omega_s cs$

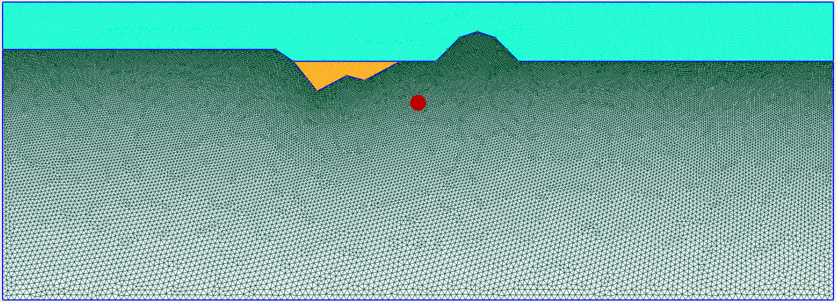

Fig. 13: Mesh of sedimentary basin

Propagation of elastic pulse in sedimentary basin and atmosphere

- **Energy transfer enhancement above sedimentary basin**

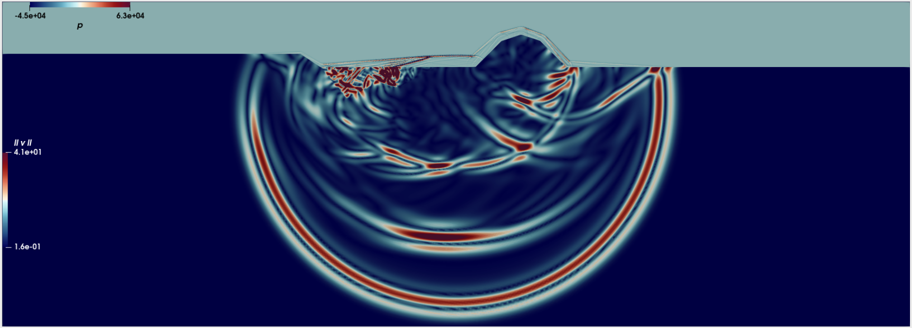

Fig. 14: Propagation of elastic pulse in sedimentary basin and atmosphere

Thank you for your attention !