



INSTITUT  
POLYTECHNIQUE  
DE PARIS



# Hybrid High-Order Methods for the Numerical Simulation of Elasto-Acoustic Wave Propagation

Romain Mottier<sup>†‡§</sup>

PhD in Applied Mathematics

**Advisors:** Alexandre Ern<sup>†‡</sup>, Laurent Guillot<sup>§</sup>

---

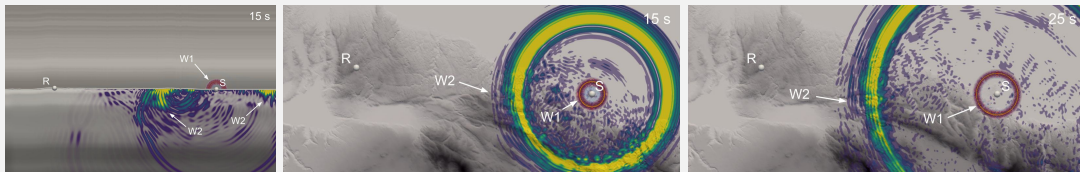
<sup>†</sup> CERMICS, Ecole des Ponts, F-77455 Marne la Vallée cedex 2

<sup>‡</sup> SERENA Project-Team, INRIA Paris, F-75647 Paris France

<sup>§</sup> CEA, DAM, DIF, F-91297 Arpajon, France

# Context & motivations

- **Mechanical waves: Acoustic (fluid) / Elastic (solid)**
- Applications: **Geophysics**, Medical imaging (elastography), Non-destructive testing etc...
- Example: Seismic monitoring of Beyrouth explosion (@Gaël Burgos - CEA/DAM/DASE)



- Objectives: **Detection, Localization, Source characterization**

- **Goal:** Simulate accurately elasto-acoustic waves propagation in **heterogeneous** and **complex** domains

- ▶ **System of the form:**

$$\mathcal{M}\ddot{\mathbf{U}} + \mathcal{K}\mathbf{U} = \mathbf{F} \quad \text{OR}$$

$$\begin{bmatrix} \mathcal{M}_{\mathbf{U}} & 0 \\ 0 & \mathcal{M}_{\Sigma} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}} \\ \dot{\Sigma} \end{bmatrix} + \begin{bmatrix} \mathcal{K}_{\mathbf{U}\mathbf{U}} & \mathcal{K}_{\mathbf{U}\Sigma} \\ \mathcal{K}_{\Sigma\mathbf{U}} & \mathcal{K}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \Sigma \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F} \end{bmatrix}$$

- ▶ **Discretization: Hybrid High-Order methods** in space & **Runge–Kutta schemes** in time

# Table of Contents

## I - State of the art

## II - HHO for elasto-acoustic coupling



R. Mottier, A. Ern, R. Khot, L. Guillot (2025). Accepted in M2AN

*Hybrid high-order methods for elasto-acoustic wave propagation in the time domain*



R. Mottier, A. Ern, L. Guillot (2025). Submitted to CMAME

*Elasto-acoustic wave propagation in geophysical media using hybrid high-order methods on general meshes*

## III - HHO on unfitted meshes: stabilization by polynomial extension



E. Burman, A. Ern, R. Mottier (2025). Submitted to SINUM

*Unfitted hybrid high-order methods stabilized by polynomial extension for elliptic interface problems*

## IV - Conclusion and perspectives

# Table of Contents

## I - State of the art

I.1 Numerical methods for wave propagation

I.2 Hybrid nonconforming methods: HDG/HHO

I.3 Further insight into HHO

# I.1. Numerical methods for wave propagation

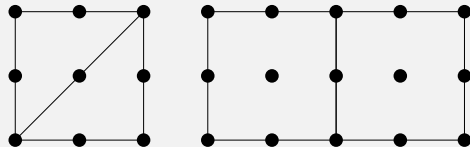
■ **Finite Differences (FD)**: Ideal for simple geometries  [van Vossen, Robertsson, Chapman \(2002\)](#)

■ **Finite Element Methods (cG)**  [Marfurt \(1984\)](#)  [Basabe, Sen \(2007\)](#)

✓ Allow for unstructured meshes

✗ Non-local polynomial bases

✗ Non-diagonal mass matrix:  
**Mass lumping** required for explicit schemes



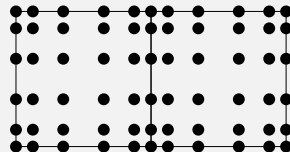
■ **Spectral Element Method (SEM)**  [Patera \(1984\)](#)  [Komatitsch, Vilotte \(1998\)](#)  [Cohen \(2002\)](#)

► Quadrature points: tensorized GLL nodes

✓ Diagonal mass matrix

✓ Parallelizable and scalable for HPC applications

✗ Mainly quadrangular/hexahedral meshes



## ■ Main issue: Complex meshes for realistic geological structures

- ▶ Hard to generate meshes with only one simple shapes
- ▶ Meshes can include hanging nodes (*e.g.* independent subdomains meshing, etc...)

## ■ Solution: Polytopal discretization methods

QUADRANGLES



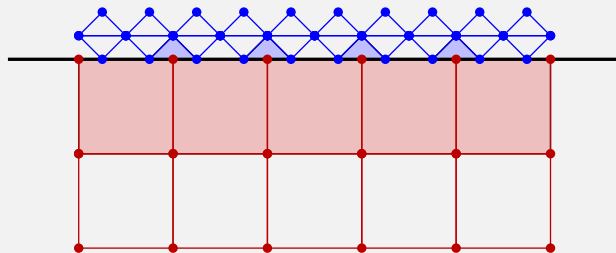
HEXAGONS



TRIANGLES



QUADRANGLES



## ■ Two possibilities:

- ▶  $H^1$ -conforming polytopal methods: **Virtual Element Methods (VEM)**
- ▶ Nonconforming methods: **class of discontinuous Galerkin methods**

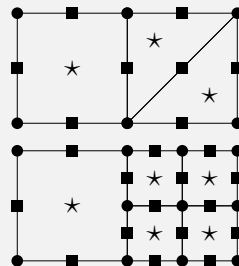
## ■ Virtual Element Methods (VEM)

📄 da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo (2013)

📄 Dassi, Fumagalli, Mazzieri, Scotti, Vacca (2022)

📄 Wriggers, Junker (2024)

✗ Same issue as cG: **Non-diagonal mass matrix**  
(dofs attached to low-dimensional mesh entities)



## ■ Discontinuous Galerkin methods (dG)

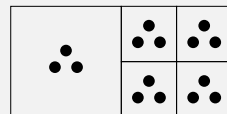
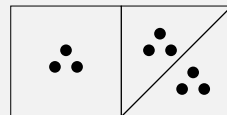
📄 Baker (1976)    📄 Wheeler (1978)    📄 Arnold (1982)

📄 Grote, Schneebeli, Schötzau (2006)

📄 Antonietti, Bonaldi, Mazzieri (2020)

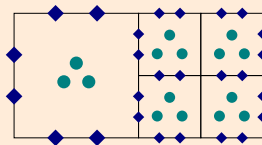
✓ Block-diagonal mass matrix    ✗ More dofs than cG

✓ Compact stencil (HPC)    ✗ **Stabilization:** Face-based  
(Minimal value)



# I.2. Hybrid Nonconforming Methods HDG/HHO

Idea: Additional unknowns attached to mesh faces



● CELL DOFS (DEGREE  $k'$ )

◆ FACE DOFS (DEGREE  $k$ )

## Hybridizable discontinuous Galerkin (HDG)

## Hybrid High-Order (HHO)

### ■ Seminal papers:

📄 Cockburn, Gopalakrishnan, Lazarov (2009)

📄 Di Pietro, Ern, Lemaire (2014)

📄 Di Pietro, Ern (2015)

### ■ Wave propagation:

📄 Stanglmeier, Nguyen, Peraire, Cockburn (2016)

📄 Burman, Duran, Ern, Steins (2021)

📄 Kronbichler, Schoeder, Müller, Wall (2016)

📄 Burman, Duran, Ern (2022a)

📄 Barucq, Rouxelin, Tordeux (2023)

📄 Steins, Ern, Jamond, Drui (2023)

📄 Ern, Steins (2024)

### ■ HHO $\equiv$ HDG but with different viewpoints

- ▶ Approximate  $(-\nabla u, u, u|_{\mathcal{F}})$
- ▶ Gradient is part of unknowns
- ▶ Stabilization in numerical flux

- ▶ Approximate  $\hat{u} := (u, u|_{\mathcal{F}})$
- ▶ Explicit gradient reconstruction
- ▶ Explicit stabilization

HHO  $\equiv$  HDG  $\equiv$  WG  $\equiv$  ncVEM

📄 Cockburn, Di Pietro, Ern (2016)

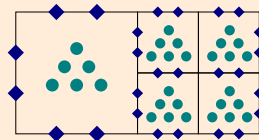
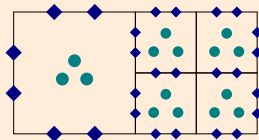
📄 Lemaire (2021)

# I.3. Further insight into HHO

- **Local degrees of freedom:**    ▶ Cell unknowns:  $u_T \in P^{k'}(T)$     ▶ Face unknowns:  $u_{\partial T} \in P^k(\partial T)$

EQUAL-ORDER:  $k' = k$  ( $= 1$ )

MIXED-ORDER:  $k' = k + 1$  ( $= 2$ )



● CELL DOFS (DEGREE  $k'$ )

◆ FACE DOFS (DEGREE  $k$ )

**Local HHO unknown:**  $\hat{u}_T := (u_T, u_{\partial T}) \in \hat{P}_T^k := P^{k'}(T) \times P^k(\partial T)$

- **Global degrees of freedom:**  $\hat{u}_{\mathcal{M}} := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \hat{P}_{\mathcal{M}}^k := P_{\mathcal{T}}^{k'} \times P_{\mathcal{F}}^k$

▶ Cell unknowns:  $u_{\mathcal{T}} \in P_{\mathcal{T}}^{k'} := \bigtimes_{T \in \mathcal{T}} P^{k'}(T)$

▶ Face unknowns:  $u_{\mathcal{F}} \in P_{\mathcal{F}}^k := \bigtimes_{F \in \mathcal{F}} P^k(F)$

- Algebraic realization for elliptic problem:  $(\kappa \nabla u, \nabla w)_\Omega = (f, w)_\Omega$

$$\begin{bmatrix} \mathcal{K}_\mathcal{T} & \mathcal{K}_{\mathcal{T}\mathcal{F}} \\ \mathcal{K}_{\mathcal{F}\mathcal{T}} & \mathcal{K}_\mathcal{F} \end{bmatrix} \begin{bmatrix} \mathbf{U}_\mathcal{T} \\ \mathbf{U}_\mathcal{F} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\mathcal{T} \\ 0 \end{bmatrix}$$

- Coupling between cell and face unknowns:  $\mathcal{K}_\mathcal{F} \mathbf{U}_\mathcal{F} = -\mathcal{K}_{\mathcal{F}\mathcal{T}} \mathbf{U}_\mathcal{T}$

- Static condensation: Elimination of cell unknowns



## ■ Design of local operators

- **Gradient reconstruction**  $(\nabla u)|_T \approx \mathbf{g}_T(\hat{u}_T) \in \mathbf{P}^k(T)$ : design mimics **integration by parts**

$$\forall \hat{u}_T \in \hat{P}_T^k \quad (\mathbf{g}_T(\hat{u}_T), \mathbf{q})_T = (\nabla u_T, \mathbf{q})_T - (u_T - u_{\partial T}, \mathbf{q} \cdot \mathbf{n}_T)_{\partial T}, \quad \forall \mathbf{q} \in \mathbf{P}^k(T)$$

- **Stabilization operator**: Weakly enforces matching of trace of cell dofs with face dofs

$$S_{\partial T}(\hat{u}_T) := \begin{cases} \Pi_{\partial T}^k(u_T|_{\partial T} - u_{\partial T}) & \text{mixed-order: Lehrenfeld–Schöberl (HDG)} \\ u_T|_{\partial T} - u_{\partial T} + \text{HOC} & \text{equal-order: High-order correction} \end{cases}$$

## ■ Same advantages as dG with following improvements:

- ✓ Improved convergence rates:  $\|\cdot\|_{H^1}$ :  $\mathcal{O}(h^{k+1})$  vs.  $\mathcal{O}(h^k)$  &  $\|\cdot\|_{L^2}$ :  $\mathcal{O}(h^{k+2})$  vs.  $\mathcal{O}(h^{k+1})$
- ✓ Tuning-free and cell-based stabilization
- ✓ Attractive computational costs leveraging static condensation
- ✓ No integration needed on faces for nonlinear problems

# Table of Contents

## II - Elasto-acoustic coupling

II.1 Model problem

II.2 HHO space semi-discretization

II.3 Runge–Kutta time discretization

II.4 Numerical results



R. Mottier, A. Ern, R. Khot, L. Guillot (2025). Accepted to M2AN

*Hybrid high-order methods for elasto-acoustic wave propagation in the time domain*



R. Mottier, A. Ern, L. Guillot (2025). Submitted to CMAME

*Elasto-acoustic wave propagation in geophysical media using hybrid high-order methods on general meshes*

## II.1. Model problem

- Wave equations in first-order formulation in time  $\rightarrow$  **Wide range of efficient time integrators**

- Acoustic wave equation (in  $\Omega^F$ ):**

$$\rho^F \partial_t \mathbf{v} - \nabla p = \mathbf{0}$$

$$\frac{1}{\kappa} \partial_t p - \nabla \cdot \mathbf{v} = f^F$$

Unknowns:  $p$  pressure,  $\mathbf{v}$  velocity

Parameters:  $\rho^F, \kappa \rightarrow c_p^F := \sqrt{\frac{\kappa}{\rho^F}}$

- Elastic wave equation (in  $\Omega^S$ ):**

$$\mathbb{C}^{-1}(\lambda, \mu) \partial_t \mathbb{s} - \nabla_{\text{sym}} \mathbf{v} = \mathbf{0}$$

$$\rho^S \partial_t \mathbf{v} - \nabla \cdot \mathbb{s} = \mathbf{f}^S$$

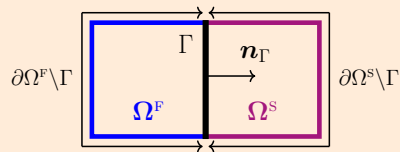
Unknowns:  $\mathbf{v}$  velocity,  $\mathbb{s}$  stress

Parameters:  $\rho^S, \mathbb{C} \rightarrow c_p^S := \sqrt{\frac{\lambda + 2\mu}{\rho^S}}, \quad c_s := \sqrt{\frac{\mu}{\rho^S}}$

- Coupling conditions (on  $\Gamma = \partial\Omega^F \cap \partial\Omega^S$ ):**

- Kinematic condition:  $\mathbf{v} \cdot \mathbf{n}_\Gamma = \mathbf{v} \cdot \mathbf{n}_\Gamma$

- Balance of forces per unit surface:  $\mathbb{s} \cdot \mathbf{n}_\Gamma = p \mathbf{n}_\Gamma$



- Initial conditions on  $(p, \mathbf{v})$  and  $(\mathbf{v}, \mathbb{s})$

- Homogeneous Dirichlet boundary conditions on  $\partial\Omega$  for simplicity

## ■ Approximation spaces

$$P^F := \{p \in H^1(\Omega^F) : p|_{\partial\Omega^F \setminus \Gamma} = 0\}, \quad M^F := L^2(\Omega^F)$$

$$\mathbf{V}^S := \{\mathbf{v} \in \mathbf{H}^1(\Omega^S) : \mathbf{v}|_{\partial\Omega^S \setminus \Gamma} = \mathbf{0}\}, \quad \mathbb{S}^S := \mathbb{L}_{\text{sym}}^2(\Omega^S)$$

■ **Weak formulation:** Find  $(\mathbf{v}, p) : (0, T_f) \rightarrow M^F \times P^F$  and  $(\mathbf{s}, \mathbf{v}) : (0, T_f) \rightarrow \mathbb{S}^S \times \mathbf{V}^S$  such that,  $\forall t \in (0, T_f)$ ,

► **Acoustic wave equations:**  $\forall (\mathbf{r}, q) \in M^F \times P^F$ ,

$$(\partial_t \mathbf{v}(t), \mathbf{r})_{\rho^F; \Omega^F} - (\nabla p(t), \mathbf{r})_{\Omega^F} = 0$$

$$(\partial_t p(t), q)_{\frac{1}{\kappa}; \Omega^F} + (\mathbf{v}(t), \nabla q)_{\Omega^F} + (\mathbf{v}(t) \cdot \mathbf{n}_\Gamma, q)_\Gamma = (f^F(t), q)_{\Omega^F}$$

► **Elastic wave equations:**  $\forall (\mathbb{b}, \mathbf{w}) \in \mathbb{S}^S \times \mathbf{V}^S$ ,

$$(\partial_t \mathbf{s}(t), \mathbb{b})_{\mathbb{C}^{-1}; \Omega^S} - (\nabla_{\text{sym}} \mathbf{v}(t), \mathbb{b})_{\Omega^S} = 0,$$

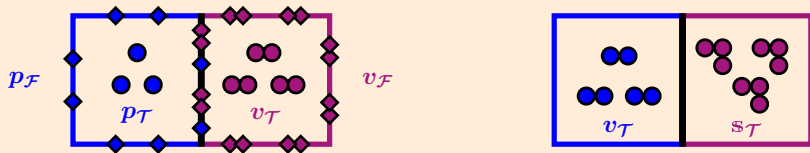
$$(\partial_t \mathbf{v}(t), \mathbf{w})_{\rho^S; \Omega^S} + (\mathbf{s}(t), \nabla_{\text{sym}} \mathbf{w})_{\Omega^S} - (p(t) \mathbf{n}_\Gamma, \mathbf{w})_\Gamma = (f^S(t), \mathbf{w})_{\Omega^S}.$$

► Skew-symmetry of differential operators

► Coupling conditions weakly imposed

■ **Main idea:** Primal variables  $(p, v) \rightarrow$  HHO discretization & Dual variables  $(v, s) \rightarrow$  dG setting

■ **Local dofs:** HHO:  $\hat{p}_T := (p_T, p_{\partial T}) \in \hat{P}_T^k$ ,  $\hat{v}_T := (v_T, v_{\partial T}) \in \hat{P}_T^k$  dG:  $v_T, s_T$



■ **Global dofs:** HHO:  $\hat{p}_{\mathcal{M}^F} := (p_{\mathcal{T}^F}, p_{\mathcal{F}^F}) \in \hat{P}_{\mathcal{M}^F}^k$ ,  $\hat{v}_{\mathcal{M}^S} := (v_{\mathcal{T}^S}, v_{\mathcal{F}^S}) \in \hat{P}_{\mathcal{M}^S}^k$  dG:  $v_{\mathcal{T}^F}, s_{\mathcal{T}^S}$

$$p_{\mathcal{T}^F} \in P_{\mathcal{T}^F}^{k'} := \bigtimes_{T \in \mathcal{T}^F} P^{k'}(T), \quad p_{\mathcal{F}^F} \in P_{\mathcal{F}^F}^k := \bigtimes_{F \in \mathcal{F}^F} P^k(F), \quad v_{\mathcal{T}^F} \in V_{\mathcal{T}^F}^k := \bigtimes_{T \in \mathcal{T}^F} P^k(T)$$

$$v_{\mathcal{T}^S} \in V_{\mathcal{T}^S}^{k'} := \bigtimes_{T \in \mathcal{T}^S} P^{k'}(T), \quad v_{\mathcal{F}^S} \in V_{\mathcal{F}^S}^k := \bigtimes_{F \in \mathcal{F}^S} P^k(F), \quad s_{\mathcal{T}^S} \in \mathbb{S}_{\mathcal{T}^S}^k := \bigtimes_{T \in \mathcal{T}^S} \mathbb{P}_{\text{sym}}^k(T)$$

## II.2. HHO space semi-discretization

- **Acoustic weak formulation (recall):** Find  $(\mathbf{v}, p) : (0, T_f) \rightarrow \mathbf{M}^F \times P^F$  such that,  $\forall t \in (0, T_f)$ ,

$$(\partial_t \mathbf{v}(t), \mathbf{r})_{(\rho^F; \Omega^F)} - (\nabla p(t), \mathbf{r})_{\Omega^F} = 0$$

$$(\partial_t p(t), q)_{(\frac{1}{\kappa}; \Omega^F)} + (\mathbf{v}(t), \nabla q)_{\Omega^F} + (\mathbf{v}(t) \cdot \mathbf{n}_\Gamma, q)_\Gamma = (f^F(t), q)_{\Omega^F}$$

- **Acoustic wave equation:** Find  $(\hat{p}_{\mathcal{M}^F}, \mathbf{v}_{\mathcal{T}^F}) : (0, T_f) \longrightarrow \hat{P}_{\mathcal{M}^F}^k \times \mathbf{V}_{\mathcal{T}^F}^k$  such that,  $\forall t \in (0, T_f)$ ,

$$(\partial_t \mathbf{v}_{\mathcal{T}^F}(t), \mathbf{r}_{\mathcal{T}^F})_{(\rho^F; \Omega^F)} - (\mathbf{g}_{\mathcal{T}^F}(\hat{p}_{\mathcal{M}^F}(t)), \mathbf{r}_{\mathcal{T}^F})_{\Omega^F} = 0$$

$$(\partial_t p_{\mathcal{T}^F}(t), q_{\mathcal{T}^F})_{(\frac{1}{\kappa}; \Omega^F)} + (\mathbf{v}_{\mathcal{T}^F}(t), \mathbf{g}_{\mathcal{T}^F}(\hat{q}_{\mathcal{M}^F}))_{\Omega^F} + s_{\mathcal{M}^F}(\hat{p}_{\mathcal{M}^F}(t), \hat{q}_{\mathcal{M}^F}) + (\mathbf{v}_{\mathcal{F}^S}(t) \cdot \mathbf{n}_\Gamma, q_{\mathcal{F}^F})_\Gamma = (f^F(t), q_{\mathcal{T}^F})_{\Omega^F}$$

- **Elastic wave equation:** Find  $(\hat{\mathbf{v}}_{\mathcal{M}^S}, \mathbb{S}_{\mathcal{T}^S}) : (0, T_f) \longrightarrow \hat{\mathbf{V}}_{\mathcal{M}^S}^k \times \mathbb{S}_{\mathcal{T}^S}^k$  such that,  $\forall t \in (0, T_f)$ ,

$$(\partial_t \mathbb{S}_{\mathcal{T}^S}(t), \mathbb{b}_{\mathcal{T}^S})_{(\mathbb{C}^{-1}; \Omega^S)} - (\mathbf{g}_{\mathcal{T}^S}^{\text{sym}}(\hat{\mathbf{v}}_{\mathcal{M}^S}(t)), \mathbb{b}_{\mathcal{T}^S})_{\Omega^S} = 0$$

$$(\partial_t \mathbf{v}_{\mathcal{T}^S}(t), \mathbf{w}_{\mathcal{T}^S})_{(\rho^S; \Omega^S)} + (\mathbb{S}_{\mathcal{T}^S}(t), \mathbf{g}_{\mathcal{T}^S}^{\text{sym}}(\hat{\mathbf{w}}_{\mathcal{M}^S}))_{\Omega^S} + s_{\mathcal{M}^S}(\hat{\mathbf{v}}_{\mathcal{M}^S}(t), \hat{\mathbf{w}}_{\mathcal{M}^S}) - (p_{\mathcal{F}^F}(t) \mathbf{n}_\Gamma, \mathbf{w}_{\mathcal{F}^S})_\Gamma = (\mathbf{f}^S(t), \mathbf{w}_{\mathcal{T}^S})_{\Omega^S}$$

- **Stabilizations act only on primal variables**

■ **Global stabilization:**

$$\triangleright s_{\mathcal{M}^F}(\hat{p}_{\mathcal{M}^F}, \hat{q}_{\mathcal{M}^F}) := \sum_{T \in \mathcal{T}^F} \tau_T^F (S_{\partial T}(\hat{p}_T), S_{\partial T}(\hat{q}_T))_{\partial T}$$

$$\triangleright s_{\mathcal{M}^S}(\hat{\mathbf{v}}_{\mathcal{M}^S}, \hat{\mathbf{w}}_{\mathcal{M}^S}) := \sum_{T \in \mathcal{T}^S} \tau_T^S (\mathbf{S}_{\partial T}(\hat{\mathbf{v}}_T), \mathbf{S}_{\partial T}(\hat{\mathbf{w}}_T))_{\partial T}$$

**Two strategies:**

$\tau_T^S = \mathcal{O}(\frac{1}{h}) = \tau_T^F$   
(standard for elliptic problem)

or

$\tau_T^S = \mathcal{O}(1) = \tau_T^F$   
(standard for hyperbolic problem)

■ **Mechanical energy:**  $\mathcal{E}_h(t) := \frac{1}{2} \|\mathbf{v}_{\mathcal{T}}(t)\|_{\rho^F; \Omega^F}^2 + \frac{1}{2} \|p_{\mathcal{T}}(t)\|_{\frac{1}{\kappa}; \Omega^F}^2 + \frac{1}{2} \|\mathbf{v}_{\mathcal{T}}(t)\|_{\rho^S; \Omega^S}^2 + \frac{1}{2} \|\mathcal{S}_{\mathcal{T}}(t)\|_{\mathbb{C}^{-1}; \Omega^S}^2$

■ **Space semi-discrete energy balance**

$$\mathcal{E}_h(t) + \int_0^t \left\{ s_{\mathcal{M}^F}(\hat{p}_{\mathcal{M}^F}(\tau), \hat{p}_{\mathcal{M}^F}(\tau)) + s_{\mathcal{M}^S}(\hat{\mathbf{v}}_{\mathcal{M}^S}(\tau), \hat{\mathbf{v}}_{\mathcal{M}^S}(\tau)) \right\} d\tau =$$

$$\mathcal{E}_h(0) + \int_0^t \left\{ (f^F(\tau), p_{\mathcal{T}^F}(\tau))_{\Omega^F} + (\mathbf{f}^S(\tau), \mathbf{v}_{\mathcal{T}^S}(\tau))_{\Omega^S} \right\} d\tau$$

## ■ Error analysis (under maximal regularity assumption)

$$\begin{aligned} & \| \mathbf{v} - \mathbf{v}_{\mathcal{T}^F} \|_{\rho^F; \Omega^F} + \| p - p_{\mathcal{T}^F} \|_{\frac{1}{\kappa}; \Omega^F} \\ + & \| \mathbb{S} - \mathbb{S}_{\mathcal{T}^S} \|_{\mathbb{C}^{-1}; \Omega^S} + \| \mathbf{v} - \mathbf{v}_{\mathcal{T}^S} \|_{\rho^S; \Omega^S} \lesssim \begin{cases} \mathcal{O}(h^{k+\frac{1}{2}}) & \text{for } \mathcal{O}(1)\text{-stabilization} \\ \mathcal{O}(h^{k+1}) & \text{for } \mathcal{O}(\frac{1}{h})\text{-stabilization \& } k' = k + 1 \end{cases} \end{aligned}$$

## ■ Key arguments

- ▶  $L^2$ -projection for HHO unknowns and HDG<sup>+</sup>-projection for dG unknowns  [Du, Sayas \(2019\)](#)
- ▶ **Error Equations:** Difference between projected exact solution and scheme equations
- ▶ **Stability:** Testing with discrete errors + using classical inequalities  
→ discrete energy on lhs and consistency error on rhs
- ▶ **Consistency error bound**

■ Use of the HDG-projection  [Cockburn, Gopalakrishnan, Sayas \(2010\)](#)

■ On simplices with  $\mathcal{O}(1)$ -stabilization  [Ern, Khot \(2024+\)](#):

$$k + \frac{1}{2} \longrightarrow k + 1$$

## ■ Space semi-discrete algebraic realization

$$\begin{bmatrix} \mathcal{M}_{\mathcal{T}^F}^{\rho^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{M}_{\mathcal{T}^F}^{\frac{1}{\kappa}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{M}_{\mathcal{T}^S}^{\mathbb{C}^{-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}_{\mathcal{T}^S}^{\rho^S} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{M}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{S}_{\mathcal{T}^S} \\ \mathbf{V}_{\mathcal{T}^S} \\ \hline \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{V}_{\mathcal{F}^S} \end{bmatrix} + \begin{bmatrix} 0 & \mathcal{G}_{\mathcal{T}^F} & 0 & 0 & \mathcal{G}_{\mathcal{T}^F \mathcal{F}^F} & 0 \\ -\mathcal{G}_{\mathcal{T}^F}^\dagger & \Sigma_{\mathcal{T}^F} & 0 & 0 & \Sigma_{\mathcal{T}^F \mathcal{F}^F} & 0 \\ 0 & 0 & 0 & \mathcal{H}_{\mathcal{T}^S} & 0 & \mathcal{H}_{\mathcal{T}^S \mathcal{F}^S} \\ 0 & 0 & -\mathcal{H}_{\mathcal{T}^S}^\dagger & \Sigma_{\mathcal{T}^S} & 0 & \Sigma_{\mathcal{T}^S \mathcal{F}^S} \\ \hline -\mathcal{G}_{\mathcal{T}^F \mathcal{F}^F}^\dagger & \Sigma_{\mathcal{T}^F \mathcal{F}^F}^\dagger & 0 & 0 & \Sigma_{\mathcal{F}^F} & \mathcal{C}_{\mathcal{F}^\Gamma} \\ 0 & 0 & -\mathcal{H}_{\mathcal{T}^S \mathcal{F}^S}^\dagger & \Sigma_{\mathcal{T}^S \mathcal{F}^S}^\dagger & -\mathcal{C}_{\mathcal{F}^\Gamma}^\dagger & \Sigma_{\mathcal{F}^S} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{S}_{\mathcal{T}^S} \\ \mathbf{V}_{\mathcal{T}^S} \\ \hline \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{V}_{\mathcal{F}^S} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^S} \\ \hline 0 \\ 0 \end{bmatrix}$$

■ **Compact formulation:**

$$\begin{bmatrix} \mathcal{M}_{\mathcal{T}} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{U}_{\mathcal{T}} \\ \mathbf{U}_{\mathcal{F}} \end{bmatrix} + \begin{bmatrix} \mathcal{K}_{\mathcal{T}} & \mathcal{K}_{\mathcal{T}\mathcal{F}} \\ \mathcal{K}_{\mathcal{F}\mathcal{T}} & \mathcal{K}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}} \\ \mathbf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathcal{T}} \\ 0 \end{bmatrix}$$

- ▶  $\mathcal{K}_{\mathcal{T}}$  is trivially block-diagonal
- ▶ **Key question:** Is  $\mathcal{K}_{\mathcal{F}}$  block-diagonal ? (**Important for explicit schemes**)

■ Structure of  $\mathcal{K}_{\mathcal{F}}$ :

$$\mathcal{K}_{\mathcal{F}} = \begin{bmatrix} \Sigma_{\mathcal{F}^F} & \mathcal{C}_{\mathcal{F}^\Gamma} \\ -\mathcal{C}_{\mathcal{F}^\Gamma} & \Sigma_{\mathcal{F}^S} \end{bmatrix} = \begin{bmatrix} \Sigma_{\mathcal{F}^{OF}} & 0 & 0 & 0 \\ 0 & \Sigma_{\mathcal{F}^{OS}} & 0 & 0 \\ 0 & 0 & \Sigma_{\mathcal{F}^\Gamma} & \mathcal{C}_{\mathcal{F}^\Gamma} \\ 0 & 0 & -\mathcal{C}_{\mathcal{F}^\Gamma} & \Sigma_{\mathcal{F}^\Gamma} \end{bmatrix}$$

$$\begin{bmatrix} \Sigma_{F_1^\Gamma} & \mathcal{C}_{F_1^\Gamma} & 0 & 0 & 0 \\ -\mathcal{C}_{F_1^\Gamma} & \Sigma_{F_1^\Gamma} & 0 & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & 0 & \Sigma_{F_n^\Gamma} & \mathcal{C}_{F_n^\Gamma} \\ 0 & 0 & 0 & -\mathcal{C}_{F_n^\Gamma} & \Sigma_{F_n^\Gamma} \end{bmatrix}$$

- $\Sigma_{\mathcal{F}^F}$  and  $\Sigma_{\mathcal{F}^S}$  block-diagonal only for **Lehrenfeld–Schöberl** and **Least-squares** stabilizations

$$S_{\partial T}(\hat{p}_T) := \begin{cases} \Pi_{\partial T}^k(p_T|_{\partial T} - p_{\partial T}) & \text{mixed-order: Lehrenfeld–Schöberl (HDG)} \\ p_T|_{\partial T} - p_{\partial T} + \mathbf{H} \times \mathbf{C} & \text{equal-order: Plain Least-squares} \end{cases}$$

- **Coupling terms at interface:** grouping acoustic and elastic dofs of each face

—►  $\mathcal{K}_{\mathcal{F}}$  is block diagonal

## II.3. Runge–Kutta time discretization

### ■ Recall

$$\begin{bmatrix} \mathcal{M}_{\mathcal{T}} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{U}_{\mathcal{T}} \\ \mathbf{U}_{\mathcal{F}} \end{bmatrix} + \begin{bmatrix} \mathcal{K}_{\mathcal{T}} & \mathcal{K}_{\mathcal{T}\mathcal{F}} \\ \mathcal{K}_{\mathcal{F}\mathcal{T}} & \mathcal{K}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}} \\ \mathbf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathcal{T}} \\ 0 \end{bmatrix}$$

### ■ $s$ -stage Runge–Kutta schemes

**SDIRK**( $s, s+1$ ) : Truncation order:  $\mathcal{O}(\Delta t^{s+1})$

|          |          |          |             |          |
|----------|----------|----------|-------------|----------|
| $c_1$    | $a_*$    | 0        | $\cdots$    | 0        |
| $c_2$    | $a_{21}$ | $a_*$    | $\ddots$    | 0        |
| $\vdots$ | $\vdots$ | $\ddots$ | $\ddots$    | $\vdots$ |
| $c_s$    | $a_{s1}$ | $\cdots$ | $a_{s,s-1}$ | $a_*$    |
|          | $b_1$    | $\cdots$ | $b_{s-1}$   | $b_s$    |

**ERK**( $s$ ) : Truncation order:  $\mathcal{O}(\Delta t^s)$

|          |          |          |             |          |
|----------|----------|----------|-------------|----------|
| $c_1$    | 0        | $\cdots$ | $\cdots$    | 0        |
| $c_2$    | $a_{21}$ | 0        | $\cdots$    | 0        |
| $\vdots$ | $\vdots$ | $\ddots$ | $\ddots$    | $\vdots$ |
| $c_s$    | $a_{s1}$ | $\cdots$ | $a_{s,s-1}$ | 0        |
|          | $b_1$    | $\cdots$ | $b_{s-1}$   | $b_s$    |

■ **SDIRK**( $s, s + 1$ ) **schemes**: For all  $i \in \{1, \dots, s\}$ ,

$$\begin{bmatrix} \mathcal{M}_{\mathcal{T}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n,i} \\ \mathbf{U}_{\mathcal{F}}^{n,i} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\mathcal{T}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n-1} \\ \mathbf{U}_{\mathcal{F}}^{n-1} \end{bmatrix} + \Delta t \sum_{j=1}^i a_{ij} \left( \begin{bmatrix} \mathbf{F}_{\mathcal{T}}^{n-1+c_j} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathcal{K}_{\mathcal{T}} & \mathcal{K}_{\mathcal{T}\mathcal{F}} \\ \mathcal{K}_{\mathcal{F}\mathcal{T}} & \mathcal{K}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n,j} \\ \mathbf{U}_{\mathcal{F}}^{n,j} \end{bmatrix} \right)$$

► Each stage can be rewritten as

$$\begin{bmatrix} \mathcal{M}_{\mathcal{T}} + a_* \Delta t \mathcal{K}_{\mathcal{T}} & a_* \Delta t \mathcal{K}_{\mathcal{T}\mathcal{F}} \\ a_* \Delta t \mathcal{K}_{\mathcal{F}\mathcal{T}} & a_* \Delta t \mathcal{K}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n,i} \\ \mathbf{U}_{\mathcal{F}}^{n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\mathcal{T}}^{n,i} \\ \mathbf{B}_{\mathcal{F}}^{n,i} \end{bmatrix}$$

■ **Usual static condensation: Cell dofs elimination**

- Global system **coupling only face dofs**
- **Cheap** local postprocessing to recover cell dofs

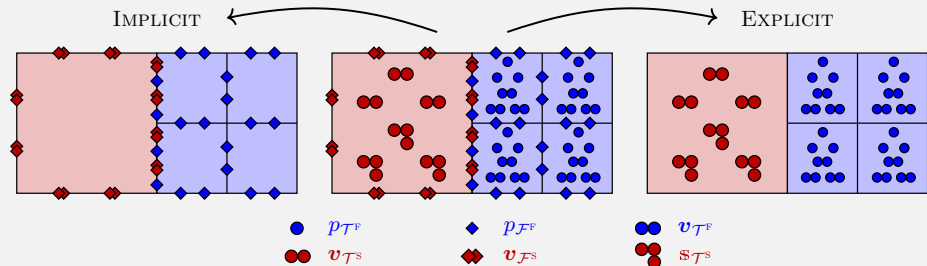
■ **ERK( $s$ ) schemes:** For all  $i \in \{1, \dots, s\}$ ,

$$\begin{bmatrix} \mathcal{M}_{\mathcal{T}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n,i} \\ \mathbf{U}_{\mathcal{F}}^{n,i} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\mathcal{T}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n-1} \\ \mathbf{U}_{\mathcal{F}}^{n-1} \end{bmatrix} + \Delta t \sum_{j=1}^{i-1} a_{ij} \left( \begin{bmatrix} \mathbf{F}_{\mathcal{T}}^{n-1+c_j} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathcal{K}_{\mathcal{T}} & \mathcal{K}_{\mathcal{T}\mathcal{F}} \\ \mathcal{K}_{\mathcal{F}\mathcal{T}} & \mathcal{K}_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}}^{n,j} \\ \mathbf{U}_{\mathcal{F}}^{n,j} \end{bmatrix} \right)$$

► **Face dofs elimination:**  $\mathcal{K}_{\mathcal{F}} \mathbf{U}_{\mathcal{F}}^{n,j} = -\mathcal{K}_{\mathcal{F}\mathcal{T}} \mathbf{U}_{\mathcal{T}}^{n,j}$

► Each stage becomes (**dG rewriting by static condensation**):

$$\mathcal{M}_{\mathcal{T}} \mathbf{U}_{\mathcal{T}}^{n,i} = \mathcal{M}_{\mathcal{T}} \mathbf{U}_{\mathcal{T}}^{n-1} + \Delta t \sum_{j=1}^{i-1} a_{ij} \left( \mathbf{F}_{\mathcal{T}}^{n-1+c_j} - (\mathcal{K}_{\mathcal{T}} - \mathcal{K}_{\mathcal{T}\mathcal{F}} \mathcal{K}_{\mathcal{F}}^{-1} \mathcal{K}_{\mathcal{F}\mathcal{T}}) \mathbf{U}_{\mathcal{T}}^{n,j} \right)$$



■ Proportion  
of face dofs:

|    |          | MIXED-ORDER |         |         | EQUAL-ORDER |         |         | $k \gg 1$      |
|----|----------|-------------|---------|---------|-------------|---------|---------|----------------|
|    |          | $k = 1$     | $k = 2$ | $k = 3$ | $k = 1$     | $k = 2$ | $k = 3$ |                |
| 2D | ACOUSTIC | 20%         | 17%     | 15%     | 25%         | 20%     | 17%     | $\frac{1}{k}$  |
|    | ELASTIC  | 22%         | 19%     | 17%     | 29%         | 23%     | 19%     | $\frac{6}{5k}$ |
| 3D | ACOUSTIC | 21%         | 19%     | 17%     | 27%         | 23%     | 20%     | $\frac{3}{2k}$ |
|    | ELASTIC  | 25%         | 23%     | 21%     | 33%         | 29%     | 25%     | $\frac{2}{k}$  |

→ Elimination of cell dofs much more efficient than face dofs

## II.4. Discretization choices

### ■ Implicit discretization

- ▶ Mixed-order discretization:
  - More unknowns but eliminated by static condensation
  - More efficient than equal-order with specific HHO stabilization  
for elliptic problems see  [Cicuttin, Ern, Pignet \(2021\)](#)
- ▶ Lehrenfeld–Schöberl  $\mathcal{O}(\frac{1}{h})$ -stabilization
  - Dual unknowns converge at  $\mathcal{O}(h^{k+1})$  in  $L^2$ -norm
  - Primal unknowns converge at  $\mathcal{O}(h^{k+2})$  in  $L^2$ -norm (superconvergence numerically observed)

### ■ Explicit discretization

- ▶ Equal-order discretization: → Less degrees of freedom
- ▶ Plain Least-Squares stabilization: → Ensure block-diagonal structure
- ▶  $\mathcal{O}(1)$ -stabilization: → Reasonable CFL condition:  $\Delta t = \mathcal{O}(h)$

## II.5. Numerical results

### ■ Academic tests cases

- ▶ CFL stability limit
- ▶ Convergences rates

### ■ Ricker wavelet in bilayered domain

- ▶ Validation by comparison to semi-analytical solution (Gar6more)
- ▶ Efficiency: Implicit vs. Explicit

### ■ Ricker wavelet in sedimentary basin

- ▶ Illustration of HHO mesh flexibility
- ▶ Comparison to a reference solver (SEM2D)

### ■ Creation of a new branch in disk++ library (<https://github.com/wareHH0use/diskpp>)

- ▶ Data structures, Discrete operators, Assembly processes, Interfacing with various solvers and Preconditioners, and Post-processing

### ■ Not discussed:

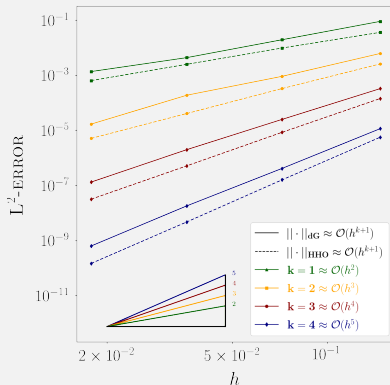
- ▶ Spectral analysis & Study of the energy dissipation
- ▶ Implementation of a 3D spectral HDG method for elastodynamics in HPC code

# Academic test case

## ■ Convergence rates on general meshes: recover theoretical rates

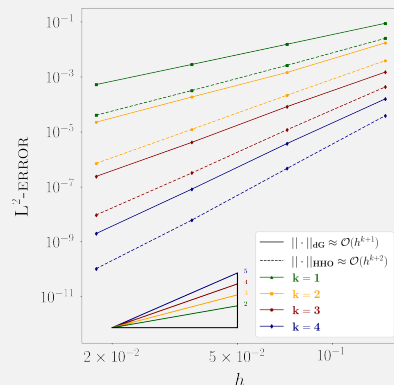
Explicit schemes with  $\mathcal{O}(1)$ -stabilization:

$$\|\cdot\|_{\text{dG}}: \mathcal{O}(h^{k+1}) \quad \|\cdot\|_{\text{HHO}}: \mathcal{O}(h^{k+1})$$



Implicit schemes with  $\mathcal{O}(\frac{1}{h})$ -stabilization:

$$\|\cdot\|_{\text{dG}}: \mathcal{O}(h^{k+1}) \quad \|\cdot\|_{\text{HHO}}: \mathcal{O}(h^{k+2})$$

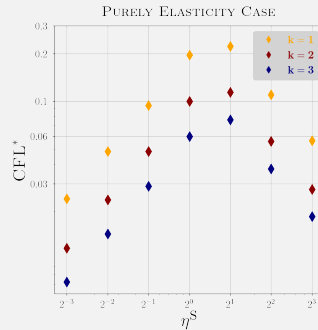
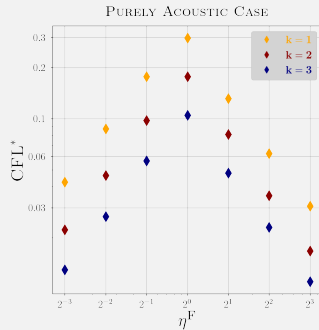


# Impact of stabilization on CFL stability limit:

$$c_{\sharp} \frac{\Delta t}{h} \leq \mathbf{CFL}^*(s, k, \eta^f, \eta^s)$$

- Weighted  $\mathcal{O}(1)$ -stabilization with  $\eta^{f/s} := 2^m$  with  $m \in \llbracket -3, 3 \rrbracket$

$$\tilde{s}_{\mathcal{M}^f}(\hat{p}_{\mathcal{M}^f}, \hat{q}_{\mathcal{M}^f}) := \eta^f s_{\mathcal{M}^f}(\hat{p}_{\mathcal{M}^f}, \hat{q}_{\mathcal{M}^f}), \quad \tilde{s}_{\mathcal{M}^s}(\hat{v}_{\mathcal{M}^s}, \hat{w}_{\mathcal{M}^s}) := \eta^s s_{\mathcal{M}^s}(\hat{v}_{\mathcal{M}^s}, \hat{w}_{\mathcal{M}^s})$$



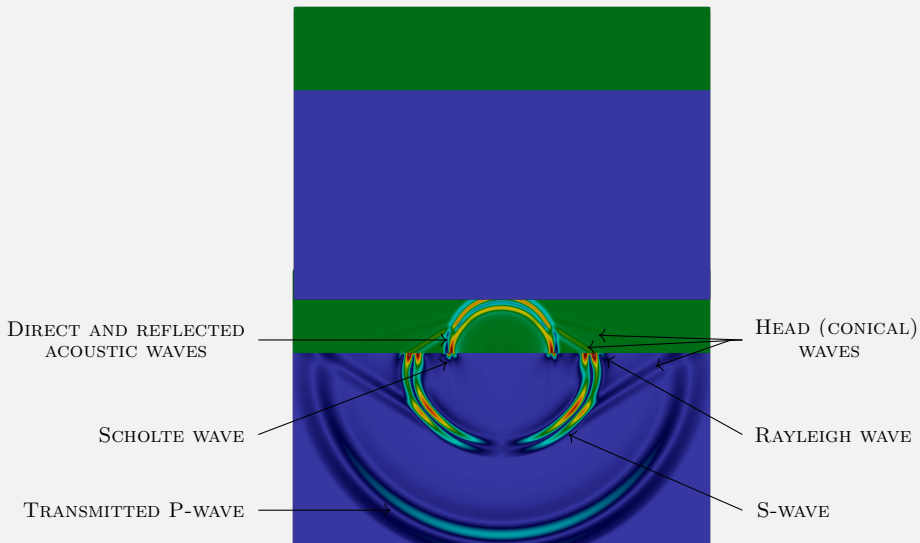
- Evolution of  $\mathbf{CFL}^*$  as  $\min(\eta, 1/\eta) \longrightarrow$  Optimal CFL for  $\eta^f \approx 1 \approx \eta^s$
- $\eta^{f/s} \gg 1 \sim \mathcal{O}(\frac{1}{h})$ -stabilization  $\longrightarrow$  degrades CFL

## ■ Influence of cell geometry

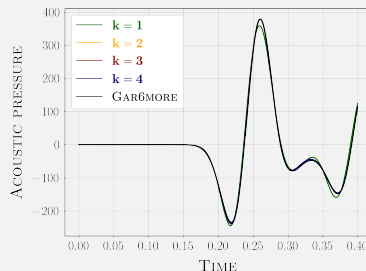
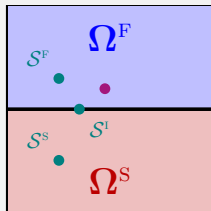
|         |       | $s = 2$     |           |            | $s = 3$     |           |            | $s = 4$     |           |            |
|---------|-------|-------------|-----------|------------|-------------|-----------|------------|-------------|-----------|------------|
| MESHERS |       | $\triangle$ | $\square$ | $\hexagon$ | $\triangle$ | $\square$ | $\hexagon$ | $\triangle$ | $\square$ | $\hexagon$ |
| $k = 1$ | CFL*  | 0.191       | 0.205     | 0.264      | 0.238       | 0.253     | 0.329      | 0.265       | 0.282     | 0.363      |
|         | RATIO | 1           | 1.07      | 1.38       | 1           | 1.06      | 1.38       | 1           | 1.06      | 1.37       |
| $k = 2$ | CFL*  | 0.106       | 0.099     | 0.136      | 0.133       | 0.123     | 0.170      | 0.147       | 0.138     | 0.188      |
|         | RATIO | 1           | 0.93      | 1.28       | 1           | 0.92      | 1.28       | 1           | 0.94      | 1.28       |
| $k = 3$ | CFL*  | 0.072       | 0.063     | 0.082      | 0.090       | 0.079     | 0.102      | 0.100       | 0.087     | 0.115      |
|         | RATIO | 1           | 0.88      | 1.13       | 1           | 0.88      | 1.13       | 1           | 0.87      | 1.15       |

- ▶ CFL\* mildly sensitive to mesh geometry
- ▶ CFL improvement for meshes with higher face counts if  $k = 1$
- ▶ Less pronounced improvement for  $k \geq 2$

# Propagation of a Ricker wavelet in a bilayered domain



## ■ Comparison to semi-analytical solution (Gar6more) Diaz, Ezziani (2008)



## ■ Efficiency study: Fair comparison (for 3D perspective)

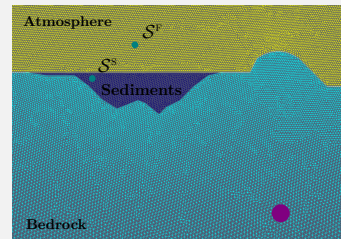
- ▶ Implicit schemes with **iterative** (BiCG) solver (with ILU preconditioner)
- ▶ Explicit schemes with **time step just below stability limit**
- ▶ Implicit time steps: **almost  $7\times$  larger**
- ▶ Implicit schemes: **decrease by 25% CPU time**
- ▶ **To be confirmed in 3D**

| SCHEMES     |         | SOLVER    | CFL*  | $c_s \frac{\Delta t}{h}$ | RATIO | CPU [s] | RATIO | ERR      |
|-------------|---------|-----------|-------|--------------------------|-------|---------|-------|----------|
| SDIRK(3, 4) | $k = 2$ | direct    | n/a   | 0.414                    | 6.90  | 1125    | 1     | 3.15e-02 |
|             | $k = 3$ |           |       | 0.414                    | 6.90  | 2284    | 2.03  | 2.52e-02 |
| SDIRK(3, 4) | $k = 2$ | iterative | n/a   | 0.414                    | 6.90  | 2228    | 1.98  | 3.91e-02 |
|             | $k = 3$ |           |       | 0.414                    | 6.90  | 4216    | 3.75  | 3.11e-02 |
| ERK(4)      | $k = 2$ | n/a       | 0.138 | 0.095                    | 1.36  | 1533    | 1.27  | 2.01e-2  |
|             | $k = 3$ |           | 0.087 | 0.060                    | 1     | 5664    | 5.03  | 1.90e-2  |

# Propagation of a Ricker wavelet in a sedimentary basin

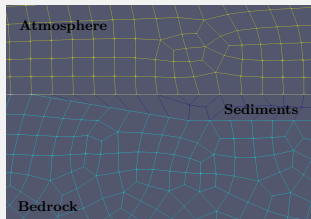
## ■ Test case setting

| MATERIAL   | $\rho^{F/S} \left[ \frac{\text{kg}}{\text{m}^3} \right]$ | $c_P^{F/S} \left[ \frac{\text{m}}{\text{s}} \right]$ | $c_P^S \left[ \frac{\text{m}}{\text{s}} \right]$ |
|------------|----------------------------------------------------------|------------------------------------------------------|--------------------------------------------------|
| ATMOSPHERE | 1.225                                                    | 343                                                  | n/a                                              |
| SEDIMENTS  | 1300                                                     | 1600                                                 | 900                                              |
| BEDROCK    | 2570                                                     | 5350                                                 | 3009                                             |

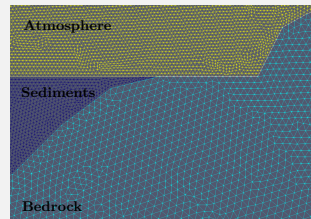


→ Illustration of HHO mesh flexibility: Typical geometry difficult to mesh in 3D

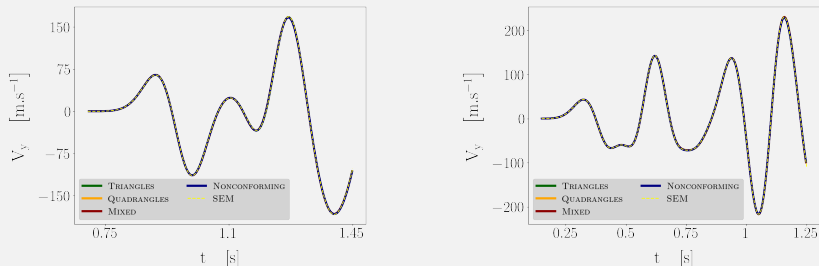
## ■ Mixed meshes



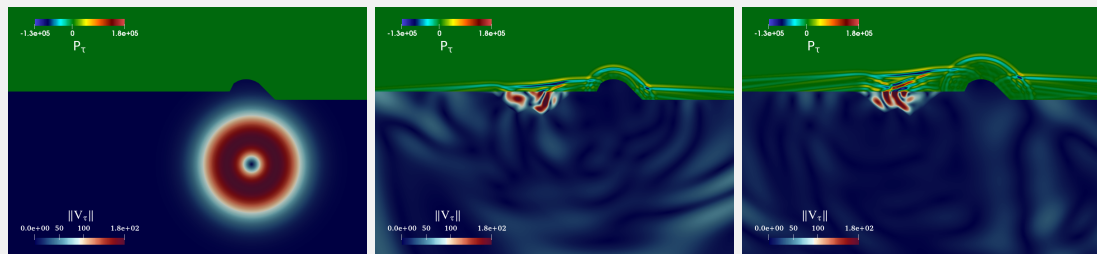
## ■ Nonconforming meshes with hanging nodes



## ■ Comparison with a reference spectral solver



## ■ Wave trapped into sedimentary basin



# Table of Contents

## III - Unfitted HHO stabilized by polynomial extension

II.1 Motivation & model problem

II.2 Unfitted FEM

II.3 Unfitted HHO

II.4 Polynomial extension

II.5 Numerical results



E. Burman, A. Ern, R. Mottier (2025). Submitted to SINUM

*Unfitted hybrid high-order methods stabilized by polynomial extension for elliptic interface problems*

## III.1. Motivation & model problem

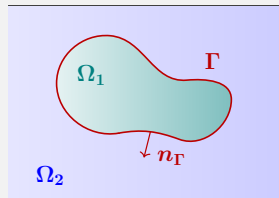
- **Goal:** Accurate simulation of elasto-acoustic waves in **heterogeneous domains with complex geometries**
- **HHO methods:**
  - ✓ Allow for efficient implicit and explicit time discretizations
  - ✓ Work on general meshes
  - ✗ Polytopal methods **work optimally on planar faces**
- ⚠ **What if interfaces cannot be meshed with planar faces**

**Idea: Use of unfitted HHO methods to handle curved interfaces**

■ First step towards wave equation: **Elliptic interface problem**

► Strong form:

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = f & \text{in } \Omega_1 \cup \Omega_2 \\ \llbracket p \rrbracket_\Gamma = g_D & \text{on } \Gamma \\ \llbracket \kappa \nabla p \rrbracket_\Gamma \cdot \mathbf{n}_\Gamma = g_N & \text{on } \Gamma \\ p = 0 & \text{on } \partial\Omega \end{cases}$$



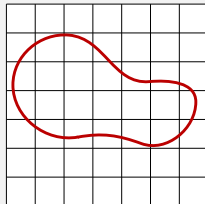
► Weak form: Find  $p \in H^1(\Omega_1 \cup \Omega_2)$ , such that  $\llbracket p \rrbracket_\Gamma = g_D$  and  $\mathbf{a}(p, q) = \ell(q)$  with

$$\bullet \quad \mathbf{a}(p, q) := \sum_{i \in \{1, 2\}} (\kappa \nabla p_i, \nabla q_i)_{\Omega_i}$$

$$\bullet \quad \ell(q) := (f, q)_\Omega + (g_N, q)_\Gamma$$

## III.2. Unfitted FEM: Basic concepts

- **Main idea: Minimize complexity of mesh generation**
  - Mesh cells arbitrarily cut by physical interfaces



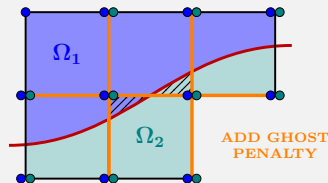
### ■ Unfitted FEM

- **Handle cut cells by doubling unknowns:** (maintain optimality)

 Hansbo, Hansbo (2002)

- **Ill-cut cell stabilization: Ghost penalty**

 Burman (2010)

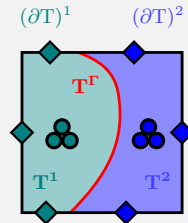


### III.3. Unfitted HHO

- **Seminal papers:**  [Burman, Ern \(2018\)](#)       [Burman, Cicuttin, Delay, Ern \(2021\)](#)
- **Different applications:** Stokes interface problems  [Burman, Delay, Ern \(2021\)](#)  
Wave propagation (second-order form)  [Burman, Duran, Ern \(2022b\)](#)

- **Doubling cell and face unknowns in cut cells:**

$$\hat{p}_T := (\hat{p}_{T^1}, \hat{p}_{T^2}) := (p_{T^1}, p_{(\partial T)^1}, p_{T^2}, p_{(\partial T)^2}) \in \hat{P}_T^k := \hat{P}_{T^1}^k \times \hat{P}_{T^2}^k$$

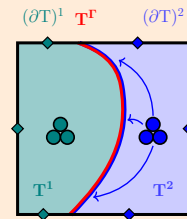


- **No unknown attached to interface**
  - **Need of mixed-order discretization:**  $k' = k + 1$

■ **Notation:**  $T^i := T \cap \Omega^i$ ,  $(\partial T)^i := \partial T \cap \Omega^i$ ,  $T^\Gamma := \partial T \cap \Gamma$

■ **Gradient reconstruction**

- Gradient is reconstructed in each sub-cell
- Jump across interface is accounted for in gradient reconstruction and close **integration by parts**



$$(g_{T^i}(\hat{p}_T), \mathbf{q})_{T^i} := (\nabla p_{T^i}, \mathbf{q})_{T^i} + (p_{(\partial T)^i} - p_{T^i}, \mathbf{q} \cdot \mathbf{n}_T)_{(\partial T)^i} - \delta_{i1}(\llbracket p_T \rrbracket_\Gamma, \mathbf{q} \cdot \mathbf{n}_\Gamma)_{T^\Gamma}$$

- **Robustness wrt contrast  $\kappa_1 \ll \kappa_2$ :** non-symmetric inclusion of  $\llbracket p_T \rrbracket_\Gamma$

## ■ Stabilization

- Usual mixed-order LS  $\mathcal{O}(\frac{1}{h})$ -stabilization (on internal faces):

$$s_{\mathcal{M}}^{\circ}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} \frac{\kappa_i}{h_T} (\Pi_{(\partial T)^i}^k(p_{T^i} - p_{(\partial T)^i}), q_{T^i} - q_{(\partial T)^i})_{(\partial T)^i}$$

- Nitsche-like penalty (at interfaces):

$$s_{\mathcal{M}}^{\Gamma}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := \sum_{T \in \mathcal{T}} \delta_{i1} \frac{\kappa_1}{h_T} (\llbracket p_T \rrbracket, \llbracket q_T \rrbracket)_{T^{\Gamma}}$$

- Total stabilization:

$$s_{\mathcal{M}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := s_{\mathcal{M}}^{\circ}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) + s_{\mathcal{M}}^{\Gamma}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}})$$

■ **Discrete problem:**  $a_{\mathcal{M}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := \ell_{\mathcal{M}}(\hat{q}_{\mathcal{M}})$

► **Discrete bilinear form:**

$$a_{\mathcal{M}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} \kappa_i(\mathbf{g}_{T^i}(\hat{p}_{\mathcal{M}}), \mathbf{g}_{T^i}(\hat{q}_{\mathcal{M}}))_{T^i} + s_{\mathcal{M}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}})$$

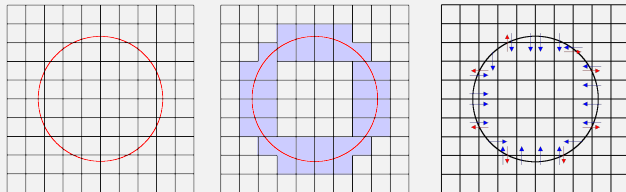
► **Discrete linear form** (with  $\ell_{\mathcal{M}}$  defined so as to ensure consistency):

$$\ell_{\mathcal{M}}(\hat{q}_{\mathcal{M}}) := \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} (f, w_{T^i})_{T^i} + \sum_{T \in \mathcal{T}^{\text{cut}}} (g_N, q_{T^2})_{T^\Gamma} + \sum_{T \in \mathcal{T}^{\text{cut}}} \kappa_1(g_D, \llbracket q_T \rrbracket h_T^{-1} - \mathbf{g}_{\mathcal{T}^1} \cdot \mathbf{n}_\Gamma)_{T^\Gamma}$$

■ **Unfitted HHO method: handling of curved interfaces in well-cut configuration**

→ Need to stabilize ill-cut configurations

# Stabilization of ill-cut cells



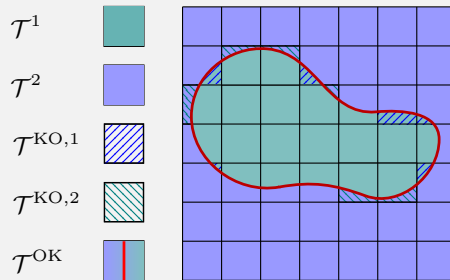
- **Cell agglomeration:**  Sollie, Bokhove, van der Vegt (2011)  Johansson, Larson (2013)
  - ✓ Leverages on polyhedral capacity of HHO methods → **No change in numerical scheme**
  - ✗ **Intrusive on mesh data structure**
- **Polynomial extension:**  This Thesis
  - ✓ **Works on initial mesh (non-intrusive on mesh data structure)**
  - ✗ Requires modification of stencil: **intrusive at assembly level**
- **Common feature:** **Pairing operator**

■ **Mesh partitioning**  $\mathcal{T} := \mathcal{T}^{\text{uncut}} \cup \mathcal{T}^{\text{cut}}$

$$\mathcal{T}^{\text{cut}} := \mathcal{T}^{\text{OK}} \cup \mathcal{T}^{\text{KO}} := \mathcal{T}^{\text{OK}} \cup \mathcal{T}^{\text{KO},1} \cup \mathcal{T}^{\text{KO},2}$$

■ **Ball condition**

- For a fixed parameter  $\vartheta \in (0, 1)$ :  $T \in \mathcal{T}^{\text{OK}} \Rightarrow T^i$  contain a ball of radius  $\vartheta h_T \quad \forall i \in \{1, 2\}$
- $\mathcal{T}^{\text{KO},1} \cap \mathcal{T}^{\text{KO},2} = \emptyset$  if mesh fine enough  [Burman, Ern \(2018\)](#)

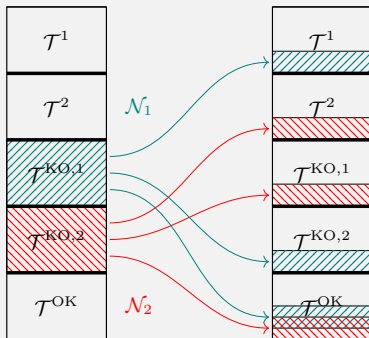


### ■ Pairing operator:

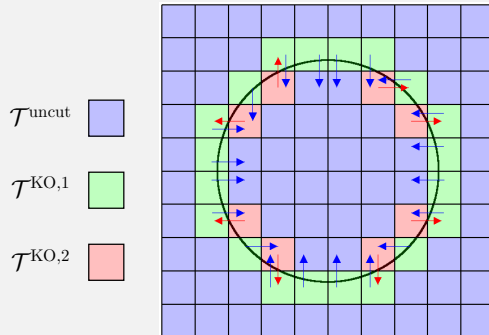
$$\mathcal{N}_i : \mathcal{T}^{\text{KO},i} \ni S \mapsto T \in (\mathcal{T}^i \cup \mathcal{T}^{\text{OK}} \cup \mathcal{T}^{\text{KO},\bar{i}}) \cap \Delta_1(S), \quad \forall i \in \{1,2\}$$

- $\Delta_1(S)$  : first layer of neighboring cells of  $S$

#### Pairing operator



#### Example



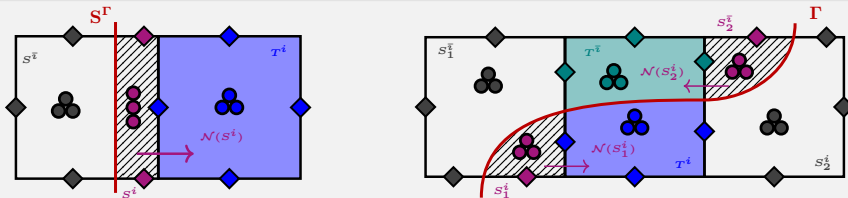
## III.4. Polynomial extension

### ■ Main idea:

- ▶ Enlarge stencil:  $\hat{p}_T^{\mathcal{N}} := (\hat{p}_T, (\hat{p}_S)_{S \in \mathcal{N}^{-1}(T)}) \in \hat{P}_T^{\mathcal{N}} := \hat{P}_T^k \times \bigtimes_{S \in \mathcal{N}^{-1}(T)} \hat{P}_S^k$
- ▶ Use ill-cut cells dofs in gradient reconstruction of **well-cut** and **uncut** (sub)cells
- ▶ Stabilize ill-cut cells dofs

### ■ Local gradient reconstruction: If subcell $T^i$ satisfies ball condition

$$\begin{aligned}
 (g_{T^i}(\hat{p}_T^{\mathcal{N}}), \mathbf{q})_{T^i} &:= (\nabla p_{T^i}, \mathbf{q})_{T^i} + (p_{(\partial T)^i} - p_{T^i}, \mathbf{q} \cdot \mathbf{n}_T)_{(\partial T)^i} - \delta_{i1} ([p_T]_{\Gamma}, \mathbf{q} \cdot \mathbf{n}_{\Gamma})_{T^{\Gamma}} \\
 &+ \sum_{S \in \mathcal{N}_i^{-1}(T)} \left\{ (p_{(\partial S)^i} - p_{S^i}, \mathbf{q}^+ \cdot \mathbf{n}_S)_{(\partial S)^i} - \delta_{i1} ([p_S]_{\Gamma}, \mathbf{q}^+ \cdot \mathbf{n}_{\Gamma})_{S^{\Gamma}} \right\}
 \end{aligned}$$



■ **Stabilization:** Lehrenfeld–Schöberl stabilization & Nitsche-like penalty at interface

► **Ill-cut stabilization:** Direct ghost penalty on skeleton

 Preuß (2018)  Lehrenfeld, Olshanskii (2019)

$$s_{\mathcal{M}}^{\mathcal{N}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := \sum_{(T,i) \in \mathcal{P}_h^{\text{OK}}} \sum_{S \in \mathcal{N}_i^{-1}(T)} \frac{\kappa_i}{h_S} (p_{S^i} - p_{T^i}^+, q_{S^i} - q_{T^i}^+)_{S^i}$$

- Connect ill-cut cell dofs with well-cut or uncut cell dofs

■ **Main error estimate:** Let  $p \in H^s(\Omega_1 \cup \Omega_2)$  with  $s \in (\frac{3}{2}, k+2]$

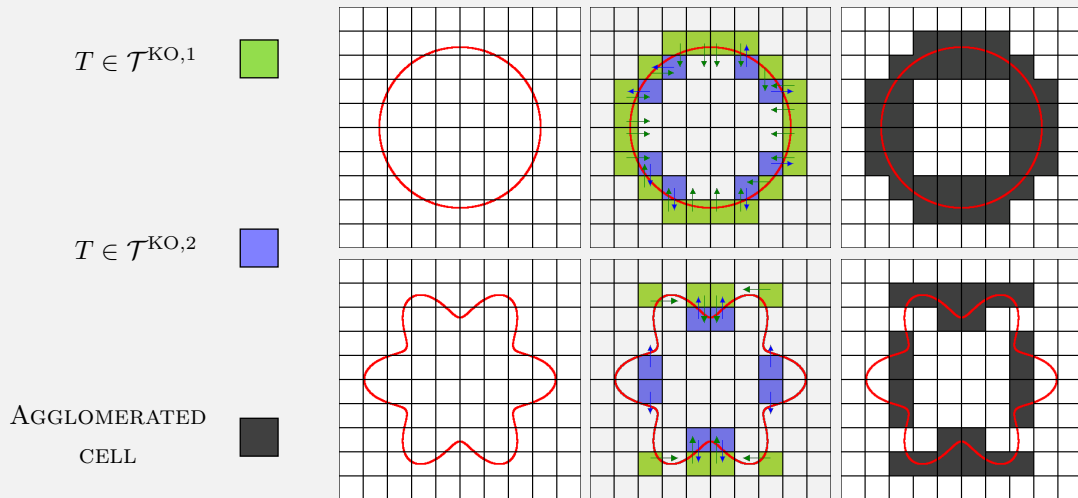
$$\left\{ \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} \kappa_i \|\nabla(p_i - p_{T^i})\|_{T^i}^2 \right\}^{\frac{1}{2}} \lesssim h^{s-1} \sum_{i \in \{1,2\}} \kappa_i^{\frac{1}{2}} |p_i|_{H^s(\Omega_i)}$$

► **Optimal convergence rates in  $H^1$ -norm:**  $O(h^{k+1})$

## III.5. Implementation details

- New branch in ProtoN C++ library with following features (porting to disk++ in progress)
  - ▶ Modal (centered and scaled) bases attached to sub-cells
  - ▶ Quadratures in cut cells based on **sub-triangulation**, using a pcw. linear approximation of interface into  $2^r$  ( $r \approx 6$ ) segments
  
- Improvement of polynomial integration in cut cells
  - ▶ Isoparametric description of interface  Lehrenfeld (2016)
  - ▶ Successive integration by parts  Antonietti, Houston, Pennesi (2018)

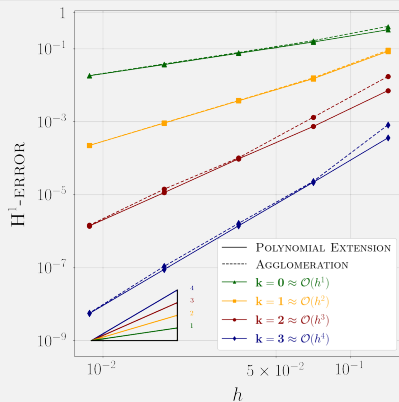
## III.6. Numerical results



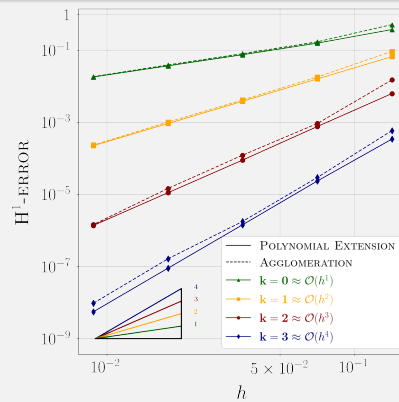
# Convergence rates on smooth solutions

$$u(x, y) = \sin(\pi x) \sin(\pi y), \quad g_D = g_N = 0, \quad \kappa_1 = \kappa_2 = 1$$

Circular interface

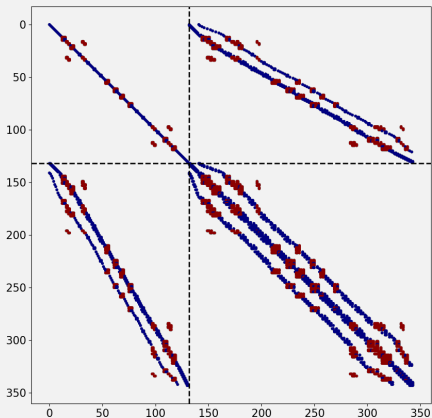


Flower-like interface

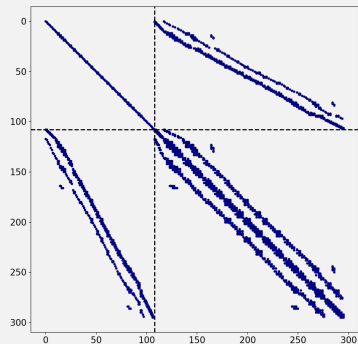


# Comparison of matrix sparsity profiles

## Polynomial extension



## Cell agglomeration

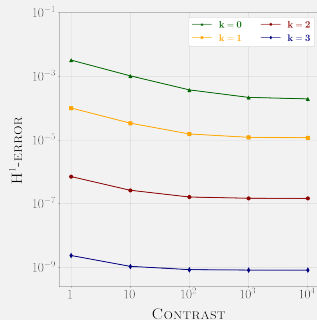


# Solutions with contrasted diffusivity

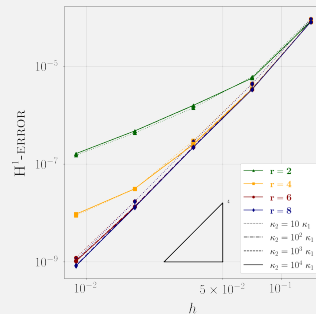
- Circular interface, no jumps ( $g_D = g_N = 0$ ), polar coordinates  $(\rho, \theta)$ :

$$u_1(\rho) = \frac{\rho^6}{\kappa_1}, \quad u_2(\rho) = \frac{\rho^6}{\kappa_2} + R^6 \left( \frac{1}{\kappa_1} - \frac{1}{\kappa_2} \right)$$

- $H^1$ -Error vs.  $\kappa_2 = 10^m \kappa_1$ ,  $r = 10$



- $H^1$ -Error vs.  $h$ ,  $k = 3$ ,  $r \in \{2, 4, 6, 8\}$ :



- Robustness wrt contrast

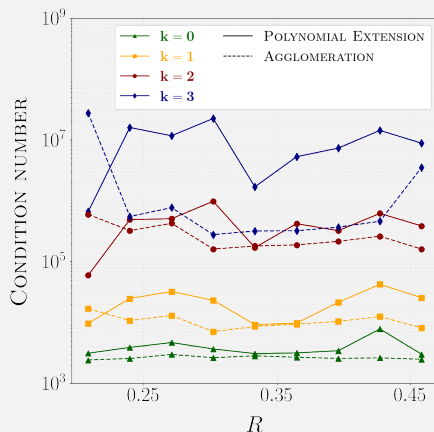
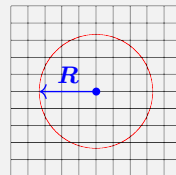
- High quality quadrature in subcell

- Same conclusions for non-polynomial jumps

# Conditioning of stiffness matrix

■ Circular interface with radius:

$$R = \frac{1}{3} + \frac{i}{32}, \quad i \in \{-4, \dots, 4\}$$



■ Grow of conditioning with polynomial degree

■ **Robust conditioning** for circular interface

# Conclusion

## Summary

- **HHO**: Accurate and robust simulation of elasto-acoustic wave propagation
  - Efficient implicit and explicit time discretizations
  - Allows for **general meshes**
- **unfitted HHO** for elliptic interface problem
  - Novel stabilization with **polynomial extension**

## Perspectives

- Radiative boundary conditions: Robin, PML etc...  [Berenger \(1994\)](#)
- Extension to other physics: linear attenuation, anisotropy, poroelasticity, etc...
- Structure-preserving RK schemes (symplectic or energy-preserving)  
 [Sánchez, Cockburn, Nguyen, Peraire \(2021\)](#)
- **unfitted HHO** for elasto-acoustic wave propagation
- **SEM-HHO coupling**

THANK YOU FOR YOUR  
ATTENTION!

# Error estimate coupling wave problem

## ■ Focus on acoustic subdomain

### ■ (1) Error equations: For all $t \in \bar{J}$ ,

$$\mathbf{N}_{\mathcal{T}^F}(t) := \mathbf{v}_{\mathcal{T}^F}(t) - \mathbf{I}_{\mathcal{T}^F}^{\text{H}+}(\mathbf{v}(t))$$

$$\hat{e}_{\mathcal{M}^F}(t) := \hat{p}_{\mathcal{M}^F}(t) - \hat{I}_{\mathcal{M}^F}^{\text{HHO}}(p(t))$$

- **Error on dual variable:** For all  $\mathbf{r}_{\mathcal{T}^F} \in \mathcal{V}_{\mathcal{T}^F}^k$ ,

$$(\partial_t \mathbf{N}_{\mathcal{T}^F}(t), \mathbf{r}_{\mathcal{T}^F})_{\rho^F; \Omega^F} - (\mathbf{G}_{\mathcal{T}^F}(\hat{e}_{\mathcal{M}^F}(t)), \mathbf{r}_{\mathcal{T}^F})_{\Omega^F} = (\partial_t \mathbf{v}(t) - \mathbf{I}_{\mathcal{T}^F}^{\text{H}+}(\partial_t \mathbf{v}(t)), \mathbf{r}_{\mathcal{T}^F})_{\rho^F; \Omega^F}$$

- **Error on primal variable:** For all  $\hat{q}_{\mathcal{M}^F} \in \hat{\mathcal{U}}_h^F$ ,

$$\begin{aligned} & (\partial_t e_{\mathcal{T}^F}(t), q_{\mathcal{T}^F})_{\frac{1}{\kappa}; \Omega^F} + (\mathbf{N}_{\mathcal{T}^F}(t), \mathbf{G}_{\mathcal{T}^F}(\hat{q}_{\mathcal{M}^F}))_{L^2(\Omega^F)} + s_{\mathcal{M}^F}(\hat{e}_{\mathcal{M}^F}(t), \hat{q}_{\mathcal{M}^F}) + (\mathbf{e}_{\mathcal{F}^S}(t) \cdot \mathbf{n}_\Gamma, q_{\mathcal{F}^F})_\Gamma \\ &= \sum_{T \in \mathcal{T}^F} ((\mathbf{v}(t) - \mathbf{I}_T^{\text{H}+}(\mathbf{v}(t))) \cdot \mathbf{n}_T, \Pi_{\partial T}^k(q_{\partial T} - q_T))_{\partial T} - s_{\mathcal{M}^F}(\hat{I}_{\mathcal{M}^F}^{\text{HHO}}(p(t)), \hat{q}_{\mathcal{M}^F}) \end{aligned}$$

# Error estimate coupling wave problem

- **(2) Stability:** Testing with discrete errors, for all  $t \in \overline{J}$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \|e_{\mathcal{T}^F}(t)\|_{\frac{1}{\kappa}; \Omega^F}^2 + \|\mathbf{N}_{\mathcal{T}^F}(t)\|_{\rho^F; \Omega^F}^2 + \dots \right\} + s_{\mathcal{M}^F}(\hat{e}_{\mathcal{M}^F}(t), \hat{e}_{\mathcal{M}^F}(t)) + \dots \\ = (\partial_t \mathbf{m}(t) - \mathbf{I}_{\mathcal{T}^F}^{\text{H}+}(\partial_t \mathbf{m}(t)), \mathbf{N}_{\mathcal{T}^F}(t))_{\rho^F; \Omega^F} + \dots + \psi_{\mathcal{M}^F}((\mathbf{m}(t), p(t)); \hat{e}_{\mathcal{M}^F}(t)) + \dots \end{aligned}$$

- **Acoustic consistency error:**

$$\begin{aligned} \psi_{\mathcal{M}^F}((\mathbf{m}(t), p(t)); \hat{q}_{\mathcal{M}^F}) := \sum_{T \in \mathcal{T}^F} ((\mathbf{m}(t) - \mathbf{I}_T^{\text{H}+}(\mathbf{m}(t))) \cdot \mathbf{n}_T, \Pi_{\partial T}^k(q_{\partial T} - q_T))_{L^2(\partial T)} \\ - s_{\mathcal{M}^F}(\hat{I}_{\mathcal{M}^F}^{\text{HHO}}(p(t)), \hat{q}_{\mathcal{M}^F}) \end{aligned}$$

# Spectral analysis

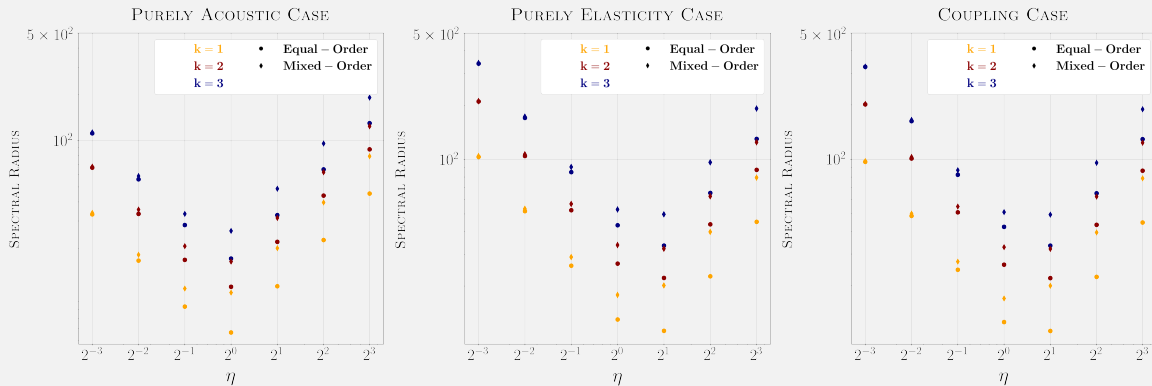


Figure 1: Spectral radius in the equal-Order and mixed-Order settings for the pure acoustic, pure linear elasticity, and elasto-acoustic coupling cases with  $k \in \{1:3\}$

|             | $k \backslash \eta/\eta_*$ | 1/8   | 1/4   | 1/2  | 1    | 2    | 4     | 8     |
|-------------|----------------------------|-------|-------|------|------|------|-------|-------|
| EQUAL-ORDER | 1                          | 55.3  | 27.8  | 14.2 | 9.9  | 19.7 | 39.5  | 78.9  |
|             | 2                          | 114.8 | 57.8  | 29.5 | 19.5 | 38.3 | 76.3  | 152.5 |
|             | 3                          | 185.2 | 93.2  | 47.7 | 29.6 | 57.3 | 113.9 | 227.3 |
| MIXED-ORDER | 1                          | 94.2  | 48.3  | 26.3 | 16.5 | 20.7 | 41.0  | 81.7  |
|             | 2                          | 195.0 | 99.3  | 53.0 | 31.8 | 33.2 | 64.7  | 128.7 |
|             | 3                          | 314.3 | 159.6 | 84.3 | 50.0 | 51.3 | 99.4  | 197.4 |

**Table 1:** Spectral radius in the equal- and mixed-order settings for the elasto-acoustic coupling with reference weights and  $k \in \{1:3\}$

|         | SIMPLICIAL MESHES $\triangle$ |             | QUADRAGULAR MESHES $\square$ |             | POLYGONAL MESHES $\hexagon$ |             |
|---------|-------------------------------|-------------|------------------------------|-------------|-----------------------------|-------------|
|         | EQUAL-ORDER                   | MIXED-ORDER | EQUAL-ORDER                  | MIXED-ORDER | EQUAL-ORDER                 | MIXED-ORDER |
| $k = 1$ | 11.6                          | 17.6        | 9.9                          | 16.5        | 10.5                        | 20.1        |
| $k = 2$ | 21.3                          | 31.4        | 19.5                         | 31.8        | 20.7                        | 37.2        |
| $k = 3$ | 33.4                          | 47.8        | 29.6                         | 50.0        | 35.2                        | 59.4        |
| $k = 4$ | 49.2                          | 69.5        | 45.3                         | 74.0        | 53.9                        | 86.6        |
| $k = 5$ | 68.0                          | 93.7        | 61.5                         | 100.6       | 76.6                        | 118.9       |
| $k = 6$ | 90.1                          | 123.2       | 83.0                         | 134.0       | 103.5                       | 156.3       |

**Table 2:** Spectral radius for different cell geometries in equal- and mixed-order settings with  $k \in \{1:6\}$  and optimal setting for  $\eta^F$  and  $\eta^S$

## Butcher tables of RK schemes

|               |               |               |
|---------------|---------------|---------------|
| $\frac{1}{4}$ | $\frac{1}{4}$ | $0$           |
| $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
|               | $\frac{1}{2}$ | $\frac{1}{2}$ |

|               |                        |               |          |
|---------------|------------------------|---------------|----------|
| $\theta$      | $\theta$               | $0$           | $0$      |
| $\frac{1}{2}$ | $\frac{1}{2} - \theta$ | $\theta$      | $0$      |
| $1 - \theta$  | $2\theta$              | $1 - 4\theta$ | $\theta$ |
|               | $\xi$                  | $1 - 2\xi$    | $\xi$    |

(1a)

SDIRK(3,4) scheme corresponds to the values  $\theta := \frac{1}{\sqrt{3}} \cos\left(\frac{\pi}{18}\right) + \frac{1}{2}$  and  $\xi := \frac{1}{6(2\theta-1)^2}$

|               |               |     |
|---------------|---------------|-----|
| $0$           | $0$           | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ |
|               | $0$           | $1$ |

|               |               |               |               |
|---------------|---------------|---------------|---------------|
| $0$           | $0$           | $0$           | $0$           |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$           | $0$           |
| $1$           | $-1$          | $2$           | $0$           |
|               | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ |

|               |               |               |               |               |
|---------------|---------------|---------------|---------------|---------------|
| $0$           | $0$           | $0$           | $0$           | $0$           |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$           | $0$           | $0$           |
| $\frac{1}{2}$ | $0$           | $\frac{1}{2}$ | $0$           | $0$           |
| $1$           | $0$           | $0$           | $1$           | $0$           |
|               | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

(1b)

# Smooth analytical solution for coupling waves

- $\Omega^F := (0, 1) \times (0, 1)$ , with  $\rho^F := 1$ ,  $\kappa := 1$ ,  $c_p^F := 1$
- $\Omega^S := (-1, 0) \times (0, 1)$ , with  $\rho^S := 1$ ,  $c_p^S := \sqrt{3}$ ,  $c_s := 1$

Analytical solution is expressed in terms of  $u$  (acoustic) and  $\mathbf{u} := (u_x, u_y)$  (elastic) so that

$$p := \partial_t u \qquad \mathbf{m} := \nabla u \qquad \text{in } \Omega^F, \quad (2a)$$

$$\mathbf{v} := \partial_t \mathbf{u} \qquad \mathbb{C}^{-1}:\mathbf{s} := \nabla_{\text{sym}} \mathbf{u} \qquad \text{in } \Omega^S. \quad (2b)$$

Source terms, (non)homogeneous Dirichlet boundary conditions, and initial conditions are defined according the following choices:

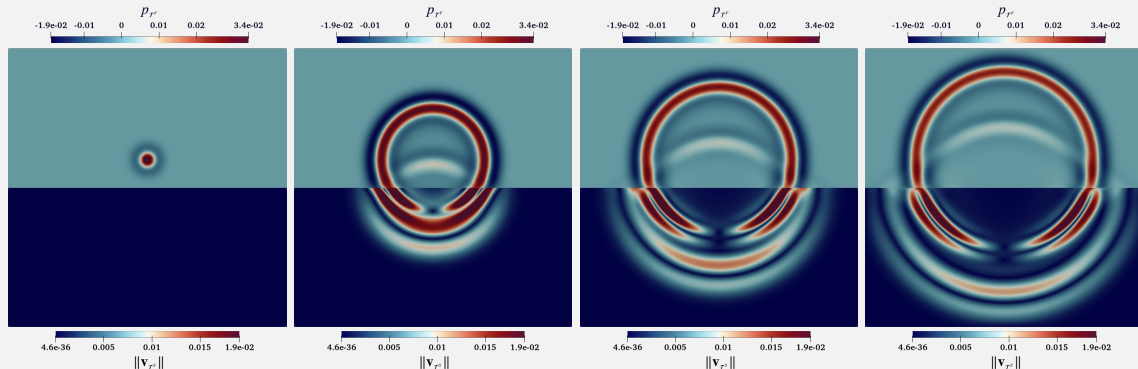
- Polynomial in space

$$u := (1 - x)x^2(1 - y)y \sin(\sqrt{2}\pi t) \qquad u_x = u_y := (1 + x)x^2(1 - y)y \sin(\sqrt{2}\pi t); \quad (3a)$$

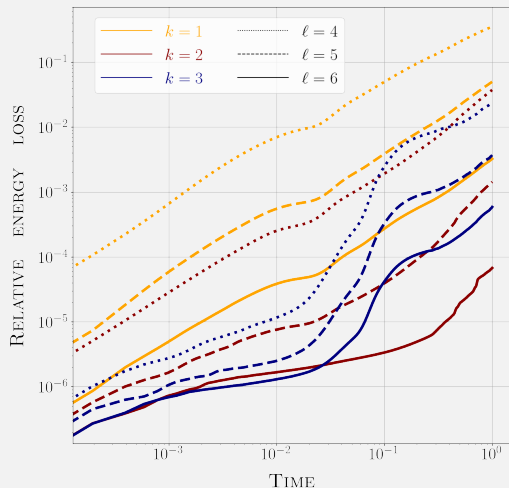
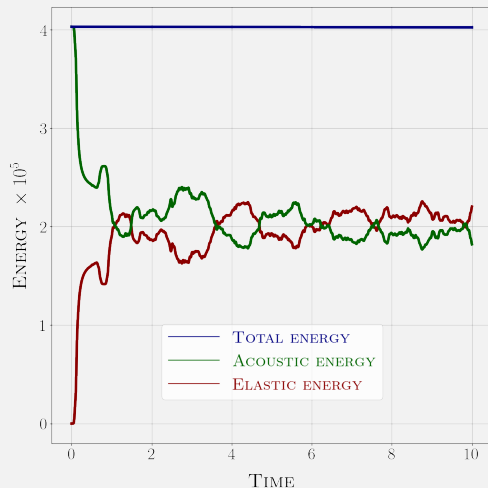
- Polynomial in time

$$u = u_x = u_y := x \sin(\pi x) \sin(\pi y) t^2. \quad (3b)$$

# Ricker wavelet in academic test case



**Figure 2:** Spatial distribution of the acoustic pressure (upper side) and the elastic velocity norm (lower side) at times  $t \in \{0, 0.25, 0.27, 0.32\}$  predicted by SDIRK(3, 4) scheme with mixed-order setting,  $\mathcal{O}(\frac{1}{h})$ -stabilization,  $k = 1$ ,  $\ell = 7$ , and  $n = 9$ .



**Figure 3:** SDIRK(3,4) scheme with  $n = 9$ . **Left:** Energy repartition as a function of the time for  $k = 3$  and  $\ell = 6$ . **Right:** Relative energy loss as function of the time for  $k \in \{1, 2, 3\}$  and  $\ell \in \{4, 5, 6\}$ .

# CFL stability limit

$$u(t, x, y) := x^2 \sin(\omega\pi x) \sin(\omega\pi y) \sin(\theta\pi t),$$

$$u_x(t, x, y) = u_y(t, x, y) := x^2 \cos(\omega\frac{\pi}{2}x) \sin(\omega\pi y) \cos(\theta\pi t),$$

- Dominant spatial evolution:  $\omega := 5$  and  $\theta := \sqrt{2}$
- Dominant temporal evolution:  $\omega := 1$  and  $\theta := 10$

---

## Algorithm 1: Bounding of the CFL Stability Limit

---

```

1 while simulation is stable do
2   for  $n = 1$  to  $N$  do
3     Compute energy  $E_n$ ;
4     Compute relative energy increase  $\Delta E := \max_n \left( \frac{|E_n - E_0|}{E_0}, \frac{|E_n - E_{n-1}|}{E_{n-1}} \right)$ ;
5     if  $\Delta E > \varepsilon$  then
6       Flag  $N$  as the first unstable time step;
7       break;
8   Flag  $N$  as the last stable time step;
9   Decrease  $N$  by  $\delta N$ ;
```

---

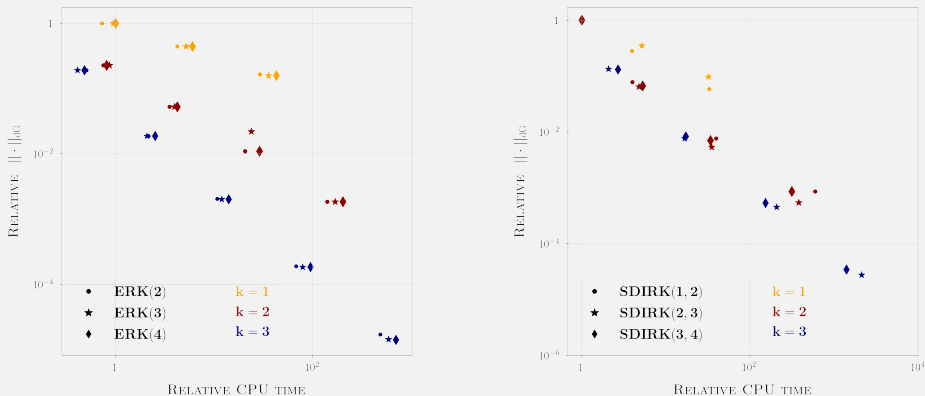
|               | $k = 1$ |         |         | $k = 2$ |         |         | $k = 3$ |         |         |
|---------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
|               | $s = 2$ | $s = 3$ | $s = 4$ | $s = 2$ | $s = 3$ | $s = 4$ | $s = 2$ | $s = 3$ | $s = 4$ |
| CFL*          | 0.205   | 0.253   | 0.282   | 0.099   | 0.123   | 0.138   | 0.063   | 0.079   | 0.087   |
| RATIO WRT $s$ | 1       | 1.23    | 1.38    | 1       | 1.24    | 1.39    | 1       | 1.25    | 1.38    |
| RATIO WRT $k$ | 1       | 1       | 1       | 0.48    | 0.49    | 0.49    | 0.31    | 0.31    | 0.31    |

Table 3: CFL\* coefficient (and ratios thereof) for ERK( $s$ ),  $s \in \{2, 3, 4\}$  and  $k \in \{1, 2, 3\}$ .

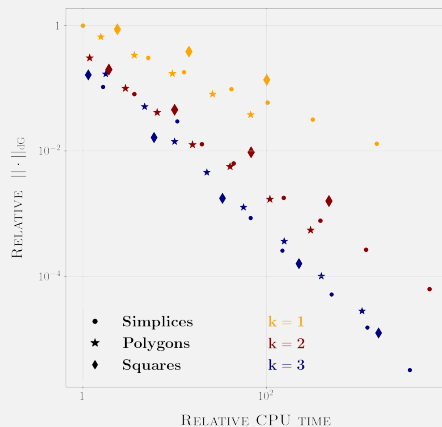
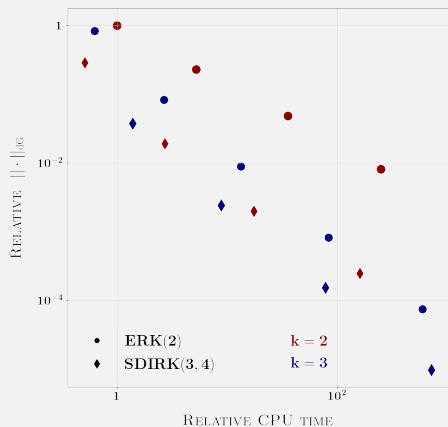
|         |       | $s = 2$     |           |            | $s = 3$     |           |            | $s = 4$     |           |            |
|---------|-------|-------------|-----------|------------|-------------|-----------|------------|-------------|-----------|------------|
| MESHERS |       | $\triangle$ | $\square$ | $\hexagon$ | $\triangle$ | $\square$ | $\hexagon$ | $\triangle$ | $\square$ | $\hexagon$ |
| $k = 1$ | CFL*  | 0.191       | 0.205     | 0.264      | 0.238       | 0.253     | 0.329      | 0.265       | 0.282     | 0.363      |
|         | RATIO | 1           | 1.07      | 1.38       | 1           | 1.06      | 1.38       | 1           | 1.06      | 1.37       |
| $k = 2$ | CFL*  | 0.106       | 0.099     | 0.136      | 0.133       | 0.123     | 0.170      | 0.147       | 0.138     | 0.188      |
|         | RATIO | 1           | 0.93      | 1.28       | 1           | 0.92      | 1.28       | 1           | 0.94      | 1.28       |
| $k = 3$ | CFL*  | 0.072       | 0.063     | 0.082      | 0.090       | 0.079     | 0.102      | 0.100       | 0.087     | 0.115      |
|         | RATIO | 1           | 0.88      | 1.13       | 1           | 0.88      | 1.13       | 1           | 0.87      | 1.15       |

Table 4: CFL\* coefficient (and ratios thereof) for ERK( $s$ ),  $s \in \{2, 3, 4\}$  and  $k \in \{1, 2, 3\}$ .

# Efficiency study

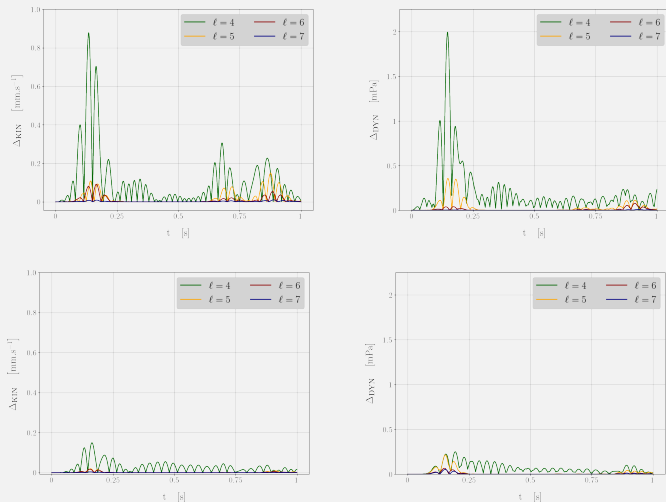


**Figure 5:** Test case with dominant spatial error. **Left panel:** Efficiency comparison for ERK( $s$ ) for  $s \in \{2, 3, 4\}$ . **Right panel:** Efficiency for SDIRK( $s, s + 1$ ) for  $s \in \{1, 2, 3\}$ .



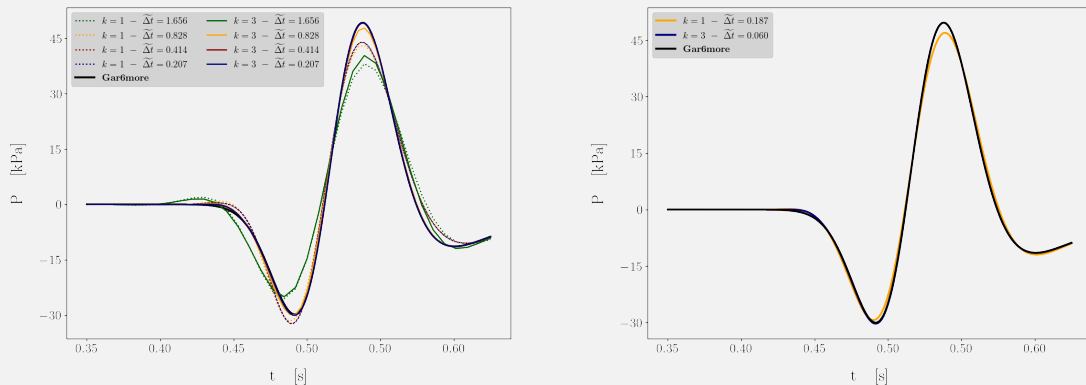
**Figure 6:** Test case with dominant spatial error. **Left panel:** Efficiency comparison for ERK(2) and SDIRK(3,4). **Right panel:** Efficiency comparison for ERK(4) on meshes composed of different cell geometries.

# Coupling condition



**Figure 7:** Errors on the coupling conditions as a function of the time predicted by SDIRK(3,4) for  $k = 1$  (top row) and  $k = 3$  (bottom row). **Left:** Kinematic errors. **Right:** Dynamic errors.

# Efficiency study on Ricker wavelet



**Figure 8:** Spatial distribution of the acoustic pressure at  $\mathcal{S}^F$  at times  $t \in [0, 0.625][s]$ . **Left column:** SDIRK(3,4),  $k \in \{1, 3\}$ , and  $\widetilde{\Delta t} \in \{0.402, 0.805, 1.609, 3.219\}$ . **Right column:** ERK(4),  $k \in \{1, 3\}$  and  $\widetilde{\Delta t} = \{0.117, 0.363\}$  (recall that  $\text{CFL}^* = 0.087$  for  $k = 3$ ).

## ⚠ What happen in ill-cut cells ?

### ■ Discrete trace inequality:

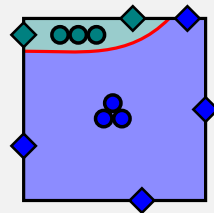
$$\|p_{T^i}\|_{(\partial T)^i} + \|p_{T^i}\|_{T^\Gamma} \lesssim C_{\text{TRACE}} h_T^{-\frac{1}{2}} \|p_{T^i}\|_{T^i},$$

►  $C_{\text{TRACE}}$  degenerate

### ■ Stability and boundedness: For all $\hat{p}_T \in \hat{P}_T$ ,

$$|||\hat{p}_T|||^2 \lesssim a_T(\hat{p}_T, \hat{p}_T) \lesssim |||\hat{p}_T|||^2 := \|\nabla p_T\|_{T^i}^2 + h_T^{-1} \|p_T - p_{\partial T}\|_{(\partial T)^i}^2$$

► Not true anymore



## ■ Stabilization:

- Keep Lehrenfeld–Schöberl stabilization & Nitsche-like penalty at interface:

$$s_{\mathcal{M}}^{\circ}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}), \quad s_{\mathcal{M}}^{\Gamma}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}})$$

- Ill-cut stabilization: **Direct ghost penalty**  Preuß (2018)  Lehrenfeld, Olshanskii (2019)

$$s_{\mathcal{M}}^{\mathcal{N}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := \sum_{(T,i) \in \mathcal{P}_h^{\text{OK}}} \sum_{S \in \mathcal{N}_i^{-1}(T)} \frac{\kappa_i}{h_S^2} (p_{S^i}^+ - p_{T^i}, q_{S^i}^+ - q_{T^i})_{T^i}$$

- Connect ill-cut cell dofs with well-cut or uncut cell dofs

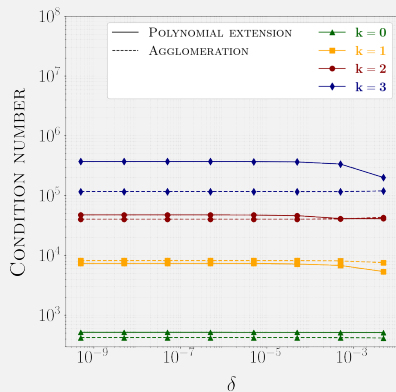
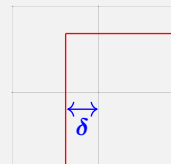
- Total stabilization:

$$s_{\mathcal{M}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) := s_{\mathcal{M}}^{\circ}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) + s_{\mathcal{M}}^{\Gamma}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}}) + s_{\mathcal{M}}^{\mathcal{N}}(\hat{p}_{\mathcal{M}}, \hat{q}_{\mathcal{M}})$$

# Conditioning of stiffness matrix

## ■ Square interface with distance to mesh:

$$\delta = 0.5 \times 10^{-p}, \quad p \in \{1, \dots, 9\}$$



- Grow of conditioning with polynomial degree
- **Robust conditioning** for severe ill-cut configurations

# Pairing criteria

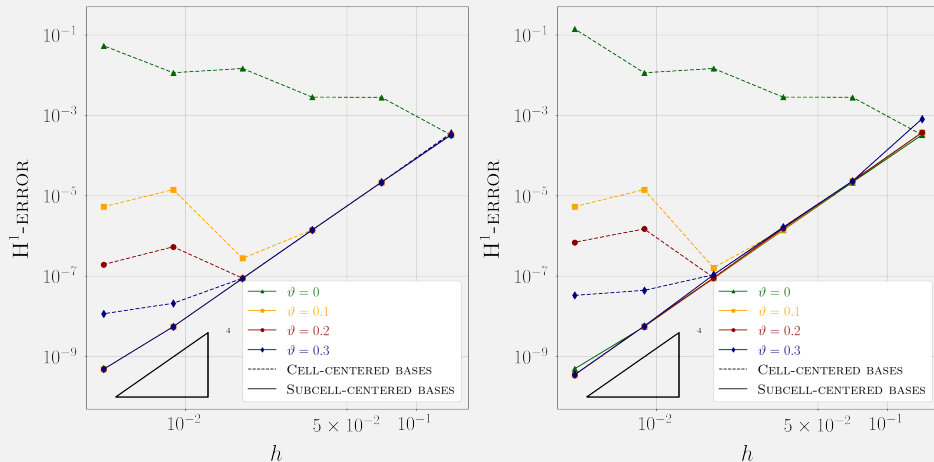


Figure 9: Errors as a function of the mesh size for the exact solution for  $k = 3$  and various values of the pairing parameter  $\vartheta$  used for flagging ill-cut cells. Left: polynomial extension. Right: cell agglomeration.

# Convergence tests with high contrast property

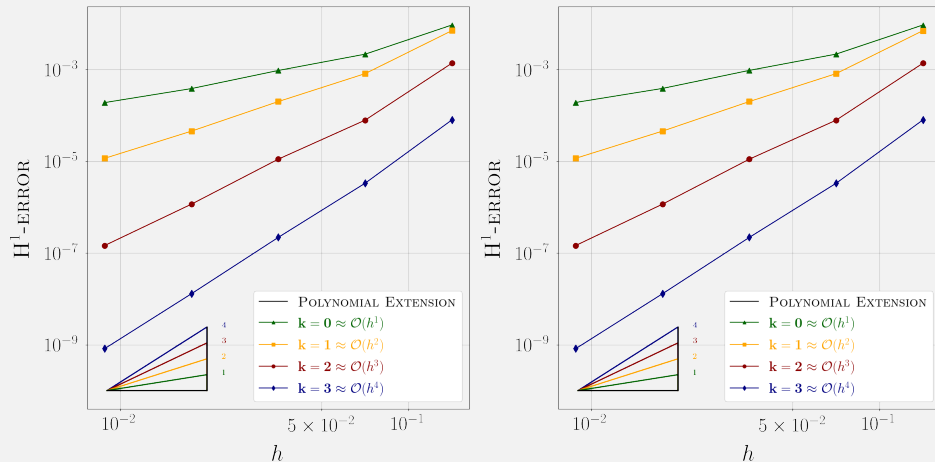
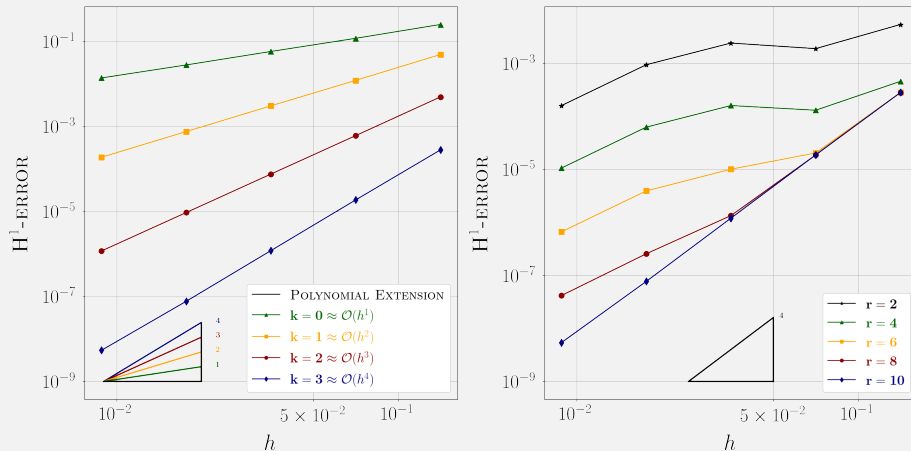


Figure 10: Errors as a function of the mesh size for the exact solution (left) and the exact solution (right) for various polynomial degrees  $k \in \{0, \dots, 3\}$ ; diffusivity contrast set to  $\kappa_2 = 10^4 \kappa_1$ .

# Convergence tests for non polynomial solution



**Figure 11:** Left: Errors as a function of the mesh size for the exact solution for various polynomial degrees  $k \in \{0, \dots, 3\}$  (with sub-triangulation parameter set to  $r = 10$ ). Right: Errors as a function of the mesh size for  $k = 3$  and  $r \in \{2, 4, 6, 8, 10\}$ . No diffusivity contrast.