

# Hybrid high-order methods for the numerical simulation of elasto-acoustic wave propagation



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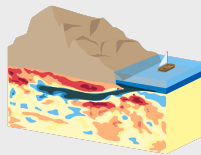
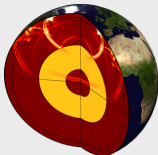
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## Goal

- Accurate simulation of seismo-acoustic waves through **heterogeneous domains with complex geometries**

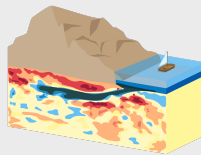
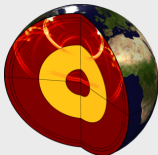


**Fig. 1:** Global seismic wave propagation **Fig. 2:** Local heterogeneities of the Earth

- Minimize numerical dispersion and dissipation for long time propagation

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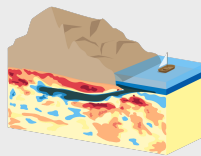
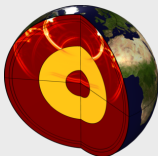
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## Commonly used numerical tools

- Spectral Element Method (cG) / Finite Differences (FD)

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## Commonly used numerical tools

- Spectral Element Method (cG) / Finite Differences (FD)
- **Main issue:** Complex mesh generation for realistic geological structures

## cG vs. dG methods

Main advantages of dG methods

- **Mesh flexibility:** Handling of **unstructured** / **polyhedral meshes**
- **Local conservativity** at the element level
- **Same order of convergence** as cG for smooth solutions:
  - ▶  $H^1$ -error:  $\mathcal{O}(h^k)$
  - ▶  $L^2$ -error:  $\mathcal{O}(h^{k+1})$

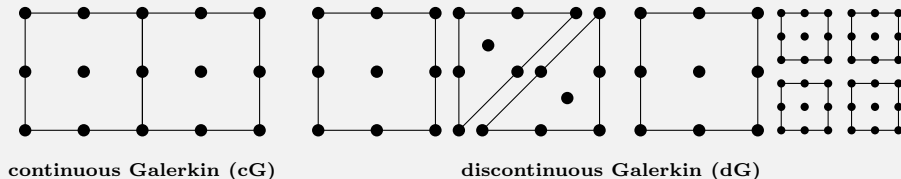
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Drawbacks of dG methods

- **Higher computational cost and memory requirement**



**Fig. 3:** Discrete unknowns for cG and dG methods



## Introduction to HHO methods

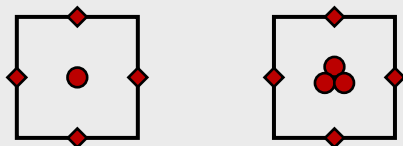
- **Seminal papers:** [Di Pietro, Ern, and Lemaire, 2014], [Di Pietro and Ern, 2015]

## Introduction to HHO methods

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### Degrees of freedom

- **Polynomial unknowns attached to mesh cells and faces**



HHO local unknowns:

$$\hat{u}_T := (u_T, u_{\partial T})$$

- Cell unknowns, degree  $k' \in \{k, k+1\}$     ◆ Face unknowns, degree  $k \geq 0$

**Fig. 4:** Local HHO unknowns. **Left:**  $k' = k = 0$ . **Right:**  $k' = k + 1 = 1$ .

- ▶ Equal-order:  $k' = k$
- ▶ Mixed-order:  $k' = k + 1$

## Design

■ **Gradient reconstruction operator:**

$$(\nabla \mathbf{u})|_T \rightarrow \mathbf{G}_T(\hat{\mathbf{u}}_T) \in \mathbb{P}^k(T; \mathbb{R}^d)$$

Design of  $\mathbf{G}_T(\hat{\mathbf{u}}_T)$  mimics an integration by parts

■ **Stabilization operator:**  $\delta_{\partial T}(\hat{\mathbf{u}}_T) := \mathbf{u}_{\partial T} - \mathbf{u}_{T|\partial T} \approx \mathbf{0}$ 

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## Advantages of HHO over dG methods

### ■ Improved error estimates for smooth solutions

▶  $H^1$ -error:  $\mathcal{O}(h^{k+1})$

▶  $L^2$ -error:  $\mathcal{O}(h^{k+2})$

### ■ Attractive computational costs

Elimination of cell unknowns by **static condensation**

- ▶ Global problem couples only face dofs
- ▶ Cell dofs recovered by local post-processing

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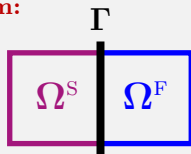
▶ Cell dofs recovered by local post-processing

## Link to other methods

$$\text{HHO} \equiv \text{HDG} \equiv \text{WG} \equiv \text{ncVEM}$$

[Cockburn, Di Pietro, and Ern, 2016]    [Lemaire, 2020]    [Cicuttin, Ern, and Pignet, 2021]

## ■ Model problem:

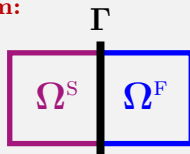


$$\Omega := \Omega^S \cup \Omega^F$$

Elasto-acoustic interface  $\Gamma$

Fig. 5: Setting for elasto-acoustic coupling

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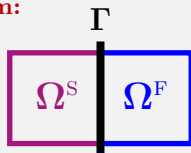
**Fig. 5:** Setting for elasto-acoustic coupling

Strong form of acoustic and elastic wave equation in 1<sup>st</sup> order formulation

$$\begin{cases} \partial_t \varepsilon - \nabla_s \mathbf{v}^S = \mathbf{0} \\ \rho^S \partial_t \mathbf{v}^S - \nabla \cdot (\mathcal{C} : \varepsilon) = \mathbf{f}^S \end{cases}$$

$$\begin{cases} \rho^F \partial_t \mathbf{v}^F - \nabla p = \mathbf{0} \\ \frac{1}{\kappa} \partial_t p - \nabla \cdot \mathbf{v}^F = f^F \end{cases}$$

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Unknowns

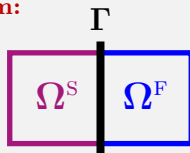
- ▶  $\mathbf{v}^S$  elastic velocity field
- ▶  $\varepsilon := \nabla_s \mathbf{u}$  linearized strain tensor

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- ▶  $p$  scalar pressure field
- ▶  $\mathbf{v}^F$  acoustic velocity field



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Unknowns

- ▶  $\mathbf{v}^S$  elastic velocity field
- ▶  $\varepsilon := \nabla_s \mathbf{u}$  linearized strain tensor

Parameters

- ▶  $\rho^S, \mathbf{C}(\lambda, \mu)$  (Lamé coefficients)
- ▶  $c_p^S := \sqrt{\frac{\lambda + 2\mu}{\rho^S}}, c_s := \sqrt{\frac{\mu}{\rho^S}}$

$$\begin{cases} \rho^F \partial_t \mathbf{v}^F - \nabla p = \mathbf{0} \\ \frac{1}{\kappa} \partial_t p - \nabla \cdot \mathbf{v}^F = f^F \end{cases}$$

- ▶  $p$  scalar pressure field
- ▶  $\mathbf{v}^F$  acoustic velocity field

- ▶  $\rho^F, \kappa$

$$\text{▶ } c_p^F := \sqrt{\frac{\kappa}{\rho^F}}$$

## Coupling conditions

$$\begin{cases} \mathbf{v}^S \cdot \mathbf{n}_\Gamma = \mathbf{v}^F \cdot \mathbf{n}_\Gamma & \blacktriangleright \text{Balance of mass + Non-penetration condition} \\ (\mathbf{C}:\boldsymbol{\varepsilon}) \cdot \mathbf{n}_\Gamma = p \mathbf{n}_\Gamma & \blacktriangleright \text{Balance of forces} \end{cases}$$

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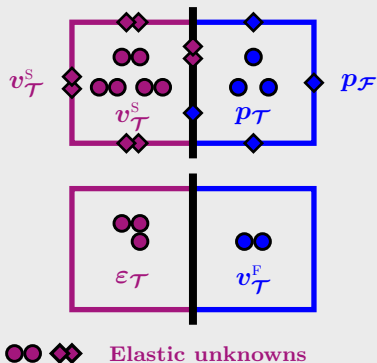
### Initial and boundary conditions

- Initial conditions on  $(\rho^S, \mathbf{v}^S)$  and  $(\rho^F, \mathbf{v}^F)$
- Homogeneous Dirichlet boundary conditions on  $\partial\Omega$  for simplicity

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## Space semi-discretization



**Fig. 6:** Elasto-acoustic unknowns with  $k' = 1$  and  $k = 0$ .

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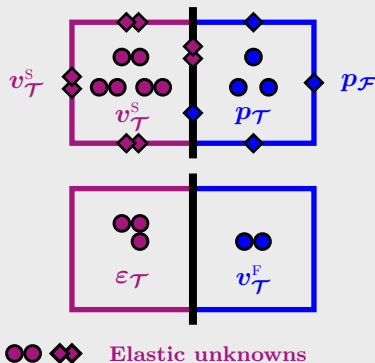


Fig. 6: Elasto-acoustic unknowns with  $k' = 1$  and  $k = 0$ .

## References

- Same discretization as for acoustic [Burman, Duran, and Ern, 2022] and elastic [Burman, Duran, Ern, and Steins, 2021] problems, but **with coupling terms**

## Local reconstruction operators

■ **Strain reconstruction:**  $\mathbf{E}_T(\hat{\mathbf{v}}_T^s) : \mathbb{P}^{k'}(T, \mathbb{R}^d) \times \mathbb{P}^k(\partial T, \mathbb{R}^d) \longrightarrow \mathbb{P}^k(T; \mathbb{R}_{\text{sym}}^{d \times d})$  s.t.

$$(\mathbf{E}_T(\hat{\mathbf{v}}_T^s), \boldsymbol{\zeta})_T = (\nabla_s \mathbf{v}_T^s, \boldsymbol{\zeta})_T - (\mathbf{v}_T^s - \mathbf{v}_{\partial T}^s, \boldsymbol{\zeta} \cdot \mathbf{n}_T)_{\partial T}, \quad \forall \boldsymbol{\zeta} \in \mathbb{P}^k(T; \mathbb{R}_{\text{sym}}^{d \times d})$$

■ **Gradient reconstruction:**  $\mathbf{G}_T(\hat{p}_T) : \mathbb{P}^{k'}(T, \mathbb{R}) \times \mathbb{P}^k(\partial T, \mathbb{R}) \longrightarrow \mathbb{P}^k(T; \mathbb{R}^d)$  s.t.

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## Local stabilization operators

■ **Mixed-order discretization:** Stabilization in HDG (Lehrenfeld-Schöberl)

$$S_{\partial T}(\hat{p}_T) := \Pi_{\partial T}^k(p_T - p_{\partial T}) \quad \mathbf{S}_{\partial T}(\hat{\mathbf{v}}_T^s) := \boldsymbol{\Pi}_{\partial T}^k(\mathbf{v}_T^s - \mathbf{v}_{\partial T}^s)$$

■ **Equal-order discretization:** Specific stabilization to HHO

- Needs additional velocity and pressure reconstructions



## HHO space semi-discretization for the elasto-acoustic coupling

■ Elastic wave equation:  $\mathbf{E}_{\mathcal{T}}(\hat{\mathbf{v}}_h^s)|_T := \mathbf{E}_T(\hat{\mathbf{v}}_T^s)$

$$(\partial_t \boldsymbol{\varepsilon}_{\mathcal{T}}(t), \mathbf{z}_{\mathcal{T}})_{\Omega^s} - (\mathbf{E}_{\mathcal{T}}(\hat{\mathbf{v}}_h^s(t)), \mathbf{z}_{\mathcal{T}})_{\Omega^s} = 0$$

$$(\rho^s \partial_t \mathbf{v}_{\mathcal{T}}^s(t), \mathbf{w}_{\mathcal{T}})_{\Omega^s} + (\mathbf{C} : \boldsymbol{\varepsilon}_{\mathcal{T}}, \mathbf{E}_{\mathcal{T}}(\hat{\mathbf{w}}_h))_{\Omega^s} + s_h^s(\hat{\mathbf{v}}_h^s, \hat{\mathbf{w}}_h) + (p_{\mathcal{F}}(t), \mathbf{w}_{\mathcal{F}} \cdot \mathbf{n}_{\Gamma})_{\Gamma} = (\mathbf{f}^s(t), \mathbf{w}_{\mathcal{T}})_{\Omega^s}$$

■ Acoustic wave equation:  $\mathbf{G}_{\mathcal{T}}(\hat{p}_h)|_T := \mathbf{G}_T(\hat{p}_T)$

$$(\rho^f \partial_t \mathbf{v}^f(t), \mathbf{r}_{\mathcal{T}})_{\Omega^f} + (\mathbf{G}_{\mathcal{T}}(\hat{p}_h(t)), \mathbf{r}_{\mathcal{T}})_{\Omega^f} = 0$$

$$(\frac{1}{\kappa} \partial_t p_{\mathcal{T}}(t), q_{\mathcal{T}})_{\Omega^f} - (\mathbf{v}^f(t), \mathbf{G}_{\mathcal{T}}(\hat{q}_h))_{\Omega^f} + s_h^f(\hat{p}_h(t), \hat{q}_h) - (\mathbf{v}^f s(t) \cdot \mathbf{n}_{\Gamma}, q_{\mathcal{F}})_{\Gamma} = (f^f(t), q_{\mathcal{T}})_{\Omega^f}$$

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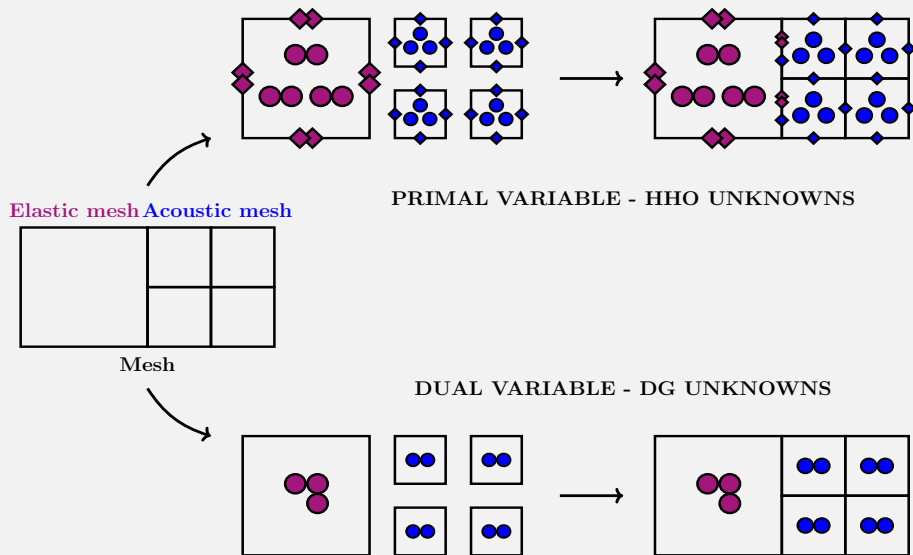
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## Global stabilization forms

$$s_h^s(\hat{\mathbf{v}}_h^s, \hat{\boldsymbol{\zeta}}_h) = \sum_{T \in \mathcal{T}_h} \tau_T^s(\mathbf{S}_{\partial T}(\hat{\mathbf{v}}_T^s), \mathbf{S}_{\partial T}(\hat{\boldsymbol{\zeta}}_T))_{\partial T}$$

$$s_h^F(\hat{p}_h, \hat{q}_h) = \sum_{T \in \mathcal{T}_h} \tau_T^F(\mathbf{S}_{\partial T}(\hat{p}_T), \mathbf{S}_{\partial T}(\hat{q}_T))_{\partial T}$$

■ with two strategies:  $\tau_T^s = \mathcal{O}(1) = \tau_T^F$  or  $\tau_T^s = \mathcal{O}(1/h) = \tau_T^F$



**Fig. 7:** Assembly procedure for the coupling in first order formulation

## Algebraic realization

### ■ Static coupling between cell and face unknowns

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & -\mathbf{G}_{\mathcal{F}} & 0 & 0 & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 & 0 & 0 \\ \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & \Sigma_{\mathcal{F}\mathcal{F}}^F & 0 & 0 & \mathbf{C}_\Gamma \\ \hline 0 & 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ 0 & 0 & -\mathbf{C}_\Gamma^\dagger & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^F \\ 0 \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^s}^s \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & -\mathbf{G}_{\mathcal{F}} & 0 & 0 & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 & 0 & 0 \\ \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & \Sigma_{\mathcal{F}\mathcal{F}}^F & 0 & 0 & \mathbf{C}_{\Gamma} \\ \hline 0 & 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^s & \Sigma_{\mathcal{T}\mathcal{F}}^s \\ 0 & 0 & -\mathbf{C}_{\Gamma}^\dagger & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^s & \Sigma_{\mathcal{F}\mathcal{F}}^s \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^F \\ 0 \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^s}^s \\ 0 \end{bmatrix}$$

### ■ Rearrangement of dofs: cell unknowns first and then face unknowns

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^s & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} + \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^s & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^s \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}_{\Gamma} \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^s & -\mathbf{C}_{\Gamma}^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^s \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^F \\ \mathbf{P}_{\mathcal{T}^F} \\ \mathbf{S}_{\mathcal{T}^s} \\ \mathbf{V}_{\mathcal{T}^s}^s \\ \mathbf{P}_{\mathcal{F}^F} \\ \mathbf{V}_{\mathcal{F}^s}^s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^F \\ 0 \\ \hline \mathbf{F}_{\mathcal{T}^s}^s \\ 0 \\ 0 \end{bmatrix}$$

## SDIRK(s, s + 1) schemes

- Generic ODE with  $f : J \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,

$$\begin{cases} y'(t) = f(t, y(t)), & \forall t \in J := [0, T) \\ y|_{t=0} = y_0 \in \mathbb{R}^m \end{cases}$$

SDIRK( $s, s + 1$ ) schemes

- Generic ODE with  $f : J \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,

$$\begin{cases} y'(t) = f(t, y(t)), & \forall t \in J := [0, T) \\ y|_{t=0} = y_0 \in \mathbb{R}^m \end{cases}$$

- SDIRK( $s, s + 1$ ) consists

- ▶ in solving sequentially for all  $i \in \{1, \dots, s\}$ ,

$$u_{\mathbf{i}}^{[n]} = u_{n-1} + \Delta t \sum_{j=1}^{\mathbf{i}} a_{ij} f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

- ▶ and setting

$$u_n := u_{n-1} + \Delta t \sum_{j=1}^s b_j f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

$c_1$	$a_{*}$	0	$\dots$	0
$c_2$	$a_{21}$	$a_{*}$	$\ddots$	0
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
$c_s$	$a_{s1}$	$\dots$	$a_{s,s-1}$	$a_{*}$
	$b_1$	$\dots$	$b_{s-1}$	$b_s$

## Algebraic realization of SDIRK-HHO

- Face-based sparse linear system to be solved at each stage
- We solve sequentially for all  $i \in \{1, \dots, s\}$ ,

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,i} \\ \mathbf{P}_{\mathcal{T}^F}^{n,i} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,i} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,i} \\ \hline \mathbf{P}_{\mathcal{F}^F}^{n,i} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n-1} \\ \mathbf{P}_{\mathcal{T}^F}^{n-1} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n-1} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n-1} \\ \hline \mathbf{P}_{\mathcal{F}^F}^{n-1} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n-1} \end{bmatrix}$$

$$+ \Delta t \sum_{j=1}^i a_{ij} \left( \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^{F,n-1+c_j} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^F}^{S,n-1+c_j} \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,j} \\ \mathbf{P}_{\mathcal{T}^F}^{n,j} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,j} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,j} \\ \hline \mathbf{P}_{\mathcal{F}^F}^{n,j} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,j} \end{bmatrix} \right)$$



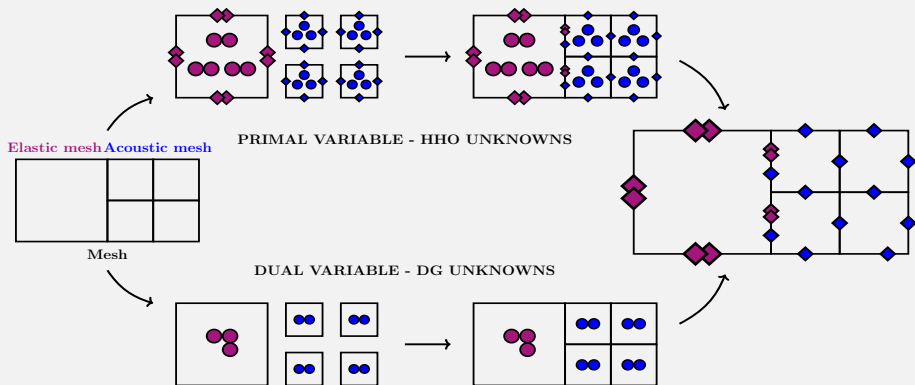
## Algebraic realization of SDIRK-HHO

- Face-based sparse linear system to be solved at each stage
- We solve sequentially for all  $i \in \{1, \dots, s\}$ ,

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,i} \\ \mathbf{P}_{\mathcal{T}^F}^{n,i} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,i} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,i} \\ \mathbf{P}_{\mathcal{F}^F}^{n,i} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n-1} \\ \mathbf{P}_{\mathcal{T}^F}^{n-1} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n-1} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n-1} \\ \mathbf{P}_{\mathcal{F}^F}^{n-1} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n-1} \end{bmatrix}$$

$$+ \Delta t \sum_{j=1}^i a_{ij} \left( \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^F}^{F,n-1+c_j} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^F}^{S,n-1+c_j} \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^F}^{F,n,j} \\ \mathbf{P}_{\mathcal{T}^F}^{n,j} \\ \hline \mathbf{S}_{\mathcal{T}^S}^{n,j} \\ \mathbf{V}_{\mathcal{T}^S}^{S,n,j} \\ \hline \mathbf{P}_{\mathcal{F}^F}^{n,j} \\ \mathbf{V}_{\mathcal{F}^S}^{S,n,j} \end{bmatrix} \right)$$

- The upper  $4 \times 4$  submatrix associated with all the cell unknowns is block-diagonal
  - Schur complement procedure



**Fig. 8:** Assembly and static condensation procedure for the coupling in first order formulation

## ERK(s) schemes

## ■ ERK(s) consists

- ▶ in updating sequentially for all  $i \in \{1, \dots, s\}$ ,

$$u_i^{[n]} = u_{n-1} + \Delta t \sum_{j=1}^{i-1} a_{ij} f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

- ▶ and setting

$$u_n := u_{n-1} + \Delta t \sum_{j=1}^s b_j f(t_{n-1} + c_j \Delta t, u_j^{[n]})$$

$c_1$	0	$\cdots$	$\cdots$	0
$c_2$	$a_{21}$	0	$\cdots$	0
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
$c_s$	$a_{s1}$	$\cdots$	$a_{s,s-1}$	0
	$b_1$	$\cdots$	$b_{s-1}$	$b_s$

## HHO-ERK scheme

■ Coupling of face unknowns at the interface  $\Gamma$ 

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^v}^{F,n,i} \\ \mathbf{P}_{\mathcal{T}^v}^{n,i} \\ \hline \mathbf{S}_{\mathcal{T}^s}^{n,i} \\ \mathbf{V}_{\mathcal{T}^s}^{S,n,i} \\ \hline \mathbf{P}_{\mathcal{F}^v}^{n,i} \\ \mathbf{V}_{\mathcal{F}^s}^{S,n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^v}^{F,n-1} \\ \mathbf{P}_{\mathcal{T}^v}^{n-1} \\ \hline \mathbf{S}_{\mathcal{T}^s}^{n-1} \\ \mathbf{V}_{\mathcal{T}^s}^{S,n-1} \\ \hline \mathbf{P}_{\mathcal{F}^v}^{n-1} \\ \mathbf{V}_{\mathcal{F}^s}^{S,n-1} \end{bmatrix}$$

$$+\Delta t \sum_{j=1}^{i-1} a_{ij} \left( \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^v}^{F,n-1+c_j} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^s}^{S,n-1+c_j} \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^v}^{F,n,j} \\ \mathbf{P}_{\mathcal{T}^v}^{n,j} \\ \hline \mathbf{S}_{\mathcal{T}^s}^{n,j} \\ \mathbf{V}_{\mathcal{T}^s}^{S,n,j} \\ \hline \mathbf{P}_{\mathcal{F}^v}^{n,j} \\ \mathbf{V}_{\mathcal{F}^s}^{S,n,j} \end{bmatrix} \right)$$

## HHO-ERK scheme

- Coupling of face unknowns at the interface  $\Gamma$

$$\begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^v}^{F,n,i} \\ \mathbf{P}_{\mathcal{T}^v}^{n,i} \\ \hline \mathbf{S}_{\mathcal{T}^s}^{n,i} \\ \mathbf{V}_{\mathcal{T}^s}^{S,n,i} \\ \hline \mathbf{P}_{\mathcal{F}^v}^{n,i} \\ \mathbf{V}_{\mathcal{F}^s}^{S,n,i} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{T}\mathcal{T}}^{v^F} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\mathcal{T}\mathcal{T}}^S & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^v}^{F,n-1} \\ \mathbf{P}_{\mathcal{T}^v}^{n-1} \\ \hline \mathbf{S}_{\mathcal{T}^s}^{n-1} \\ \mathbf{V}_{\mathcal{T}^s}^{S,n-1} \\ \hline \mathbf{P}_{\mathcal{F}^v}^{n-1} \\ \mathbf{V}_{\mathcal{F}^s}^{S,n-1} \end{bmatrix}$$

$$+ \Delta t \sum_{j=1}^{i-1} a_{ij} \left( \begin{bmatrix} 0 \\ \mathbf{F}_{\mathcal{T}^v}^{F,n-1+c_j} \\ \hline 0 \\ \mathbf{F}_{\mathcal{T}^s}^{S,n-1+c_j} \\ \hline 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\mathbf{G}_{\mathcal{T}} & 0 & 0 & -\mathbf{G}_{\mathcal{F}} & 0 \\ \mathbf{G}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^F & 0 \\ \hline 0 & 0 & 0 & -\mathbf{E}_{\mathcal{T}} & 0 & -\mathbf{E}_{\mathcal{F}} \\ 0 & 0 & \mathbf{E}_{\mathcal{T}}^\dagger & \Sigma_{\mathcal{T}\mathcal{T}}^S & 0 & \Sigma_{\mathcal{T}\mathcal{F}}^S \\ \hline \mathbf{G}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^F & 0 & 0 & \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ 0 & 0 & \mathbf{E}_{\mathcal{F}}^\dagger & \Sigma_{\mathcal{F}\mathcal{T}}^S & -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathcal{T}^v}^{F,n,j} \\ \mathbf{P}_{\mathcal{T}^v}^{n,j} \\ \hline \mathbf{S}_{\mathcal{T}^s}^{n,j} \\ \mathbf{V}_{\mathcal{T}^s}^{S,n,j} \\ \hline \mathbf{P}_{\mathcal{F}^v}^{n,j} \\ \mathbf{V}_{\mathcal{F}^s}^{S,n,j} \end{bmatrix} \right)$$

- **Key question:** how to inverse efficiently  $\begin{bmatrix} \Sigma_{\mathcal{F}\mathcal{F}}^F & \mathbf{C}^\Gamma \\ -\mathbf{C}_\Gamma^\dagger & \Sigma_{\mathcal{F}\mathcal{F}}^S \end{bmatrix}$  ?

## Rearrangement of the face terms for the inversion of coupling block

- Distinguish between internal faces in  $\Omega^S \cup \Omega^F$  and faces located on  $\Gamma$

$$\begin{bmatrix} \Sigma_{\mathcal{FF}}^F & 0 & 0 & 0 \\ 0 & \Sigma_{\mathcal{FF}}^S & 0 & 0 \\ \hline 0 & 0 & \Sigma_{\mathcal{FF}}^F & C_\Gamma \\ 0 & 0 & -C_\Gamma^\dagger & \Sigma_{\mathcal{FF}}^S \end{bmatrix} \begin{bmatrix} P_{\mathcal{F}_h^{\text{of}}} \\ V_{\mathcal{F}_h^{\text{os}}}^S \\ P_{\mathcal{F}_h^{\text{of}\Gamma}} \\ V_{\mathcal{F}_h^{\text{of}\Gamma}}^S \end{bmatrix} = \begin{bmatrix} \Sigma_{F^1}^F & C_{F^1} & 0 & 0 & 0 & 0 \\ -C_{F^1}^\dagger & \Sigma_{F^1}^S & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & & \ddots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \Sigma_{F^n}^F & C_{F^n} \\ 0 & 0 & 0 & 0 & -C_{F^n}^\dagger & \Sigma_{F^n}^S \end{bmatrix} \begin{bmatrix} P_{F^1} \\ V_{F^1}^S \\ \vdots \\ \vdots \\ P_{F^n} \\ V_{F^n}^S \end{bmatrix}$$

- Block-diagonal structure for **mixed-order** setting

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  - DG and HHO methods
  - Model Problem
- 2 RK-HHO discretization
  - HHO space semi-discretization
  - Singly diagonally implicit RK schemes
  - Explicit RK schemes
- 3 Numerical results
  - Convergence rates
  - CFL stability condition
  - Ricker wavelet
  - Sedimentary basin

### Computational parameters

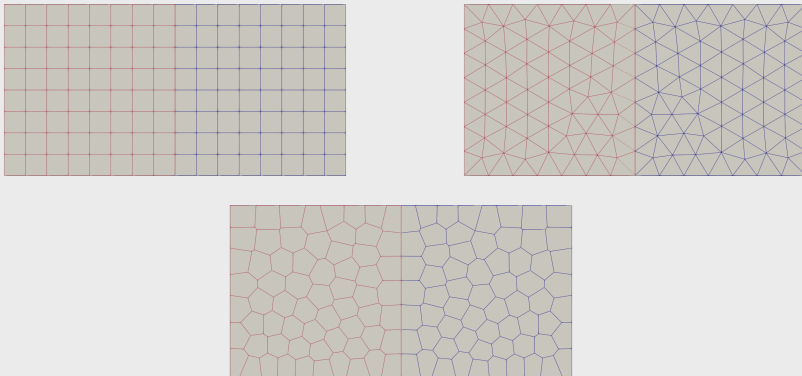
- Space refinement:  $h = 0.1 \times 2^{-\ell}$  (in each subdomain)
- Time refinement:  $\Delta t = 0.1 \times 2^{-n}$



## Computational parameters

- Space refinement:  $h = 0.1 \times 2^{-\ell}$  (in each subdomain)
- Time refinement:  $\Delta t = 0.1 \times 2^{-n}$

## Meshes



**Fig. 9:** Cartesian, simplicial and polyhedral meshes for  $\ell = 0$

## Convergence rates in space

- **Analytical solution:** polynomial in time, sinusoidal in space
- **SDIRK(3,4)-HHO scheme**      ■  $n = 8$       ■  $\ell \in \{0, 1, 2, 3, 4\}$

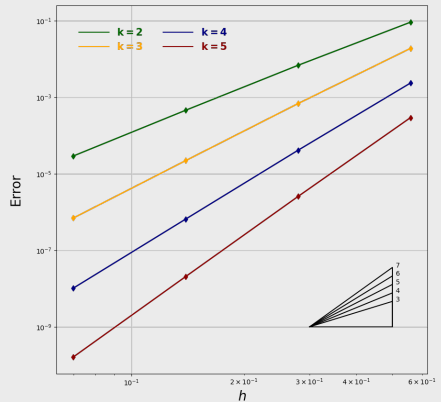
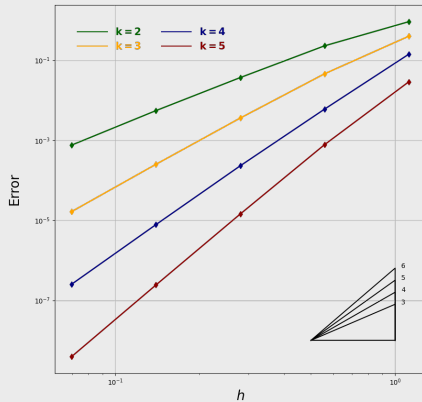
## Convergence rates in space

■ **Analytical solution:** polynomial in time, sinusoidal in space

■ **SDIRK(3,4)-HHO scheme**

■  $n = 8$

■  $\ell \in \{0, 1, 2, 3, 4\}$



**Fig. 10:**  $L^2$ -errors for the HHO-SDIRK(3,4) schemes as a function of the mesh-size. **Left:**  $\tau_T^F = \mathcal{O}(1) = \tau_T^S$ . **Right:**  $\tau_T^F = \mathcal{O}(h_T^{-1}) = \tau_T^S$

CFL stability condition for ERK( $s$ )-HHO

■ **CFL condition of the form**  $c_{\sharp} \frac{\Delta t}{h} \leq \beta(s) \mu(k) \gamma(\eta)$

- ▶  $c_{\sharp}$ : largest velocity in the domain
- ▶  $s$ : number of stages
- ▶  $k$ : polynomial order of the method
- ▶  $\eta$ : stabilisation coefficient ( $\tau_T^S, \tau_T^F \propto \eta$ )

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$\beta(s)/\beta(4)$ \ $\eta$	1/8	1/4	1/2	1	2	4	8
$s = 2$	0,72	0,72	0,72	0,72	0,72	0,72	0,72
$s = 3$	0,91	0,90	0,90	0,88	0,90	0,90	0,91

**Tab. 1:** Study of  $\beta(s)$  for  $k = 1$ ,  $\eta \in \{1/8, 1/4, \dots, 1, \dots, 4, 8\}$

CFL stability condition for ERK( $s$ )-HHO

■ **CFL condition of the form**  $c_{\sharp} \frac{\Delta t}{h} \leq \beta(s) \mu(k) \gamma(\eta)$

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$\beta(s)/\beta(4)$ \ $\eta$	1/8	1/4	1/2	1	2	4	8
$s = 2$	0,72	0,72	0,72	0,72	0,72	0,72	0,72
$s = 3$	0,91	0,90	0,90	0,88	0,90	0,90	0,91

**Tab. 1:** Study of  $\beta(s)$  for  $k = 1$ ,  $\eta \in \{1/8, 1/4, \dots, 1, \dots, 4, 8\}$

$\mu(k)/\mu(1)$ \ $\eta$	1/8	1/4	1/2	1	2	4	8
$k = 2$	0.48	0.48	0.46	0.59	0.64	0.63	0.64
$k = 3$	0.31	0.30	0.28	0.40	0.42	0.42	0.42

**Tab. 2:** Study of  $\mu(k)$  for  $s = 4$ ,  $\eta \in \{1/8, 1/4, \dots, 1, \dots, 4, 8\}$

Study of  $\gamma(\eta)$  on quadrangular meshes

■ **Meshes:** Quadrangular

■ **Computational parameters:**

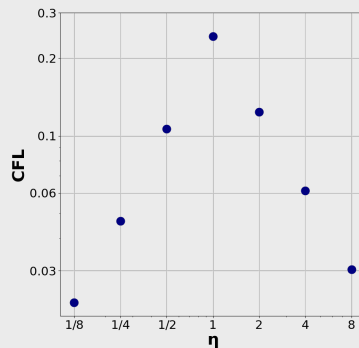
$$k = 1, \quad s = 4$$

■ **Evolution of  $\eta$  as  $\min(\eta, 1/\eta)$**

► Optimal CFL condition for  $\eta = 1$

► Stabilisation parameters

$$\tau_T^F = \mathcal{O}(h_T^{-1}) = \tau_T^S \text{ strongly} \\ \text{degrade CFL}$$



**Fig. 11:** CFL as a function of  $\eta$

Study of  $\gamma(\eta)$  on quadrangular meshes

■ **Meshes:** Quadrangular

■ **Computational parameters:**

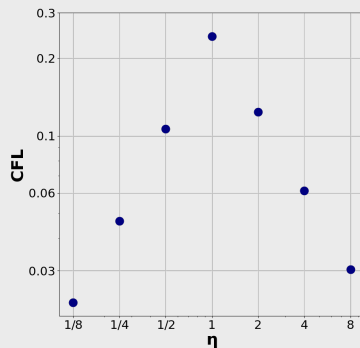
$$k = 1, \quad s = 4$$

■ **Evolution of  $\eta$  as  $\min(\eta, 1/\eta)$**

► Optimal CFL condition for  $\eta = 1$

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**Fig. 11:** CFL as a function of  $\eta$

## Influence of the cell geometry (simplex / quadrangle / polyhedron)

■ Same conclusion for  $\eta$ –dependence

■ **Slight CFL improvement with increase of number of faces**

Meshes	$\triangle$	$\square$	$\hexagon$
CFL	0.2361	0.2427	0.2670
ratio	1	1.028	1.131

**Tab. 3:** CFL coefficient for  $\eta = 1$ ,  $k = 1$ ,  $s = 4$



## Realistic test case with strong property contrast: Granite-Water

- **Initial condition:** velocity Ricker wavelet centered at point  $(x_c, y_c) \in \Omega_f$ ,

$$\mathbf{v}_0(x, y) := \theta e^{-\pi^2 \frac{r^2}{\lambda^2}} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

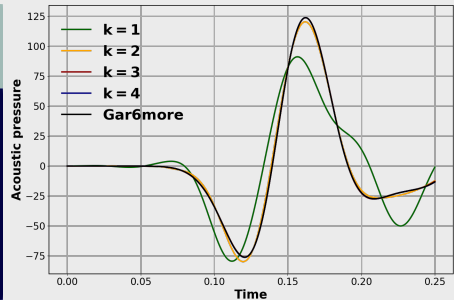
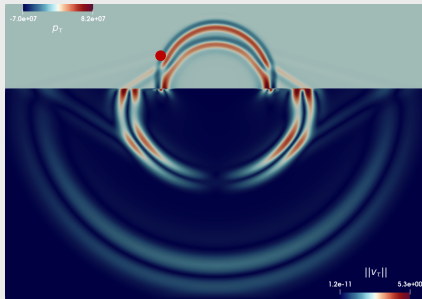
- **Material properties:** Elastic subdomain: Granite | Acoustic subdomain: Water

## Realistic test case with strong property contrast: Granite-Water

- **Initial condition:** velocity Ricker wavelet centered at point  $(x_c, y_c) \in \Omega_f$ ,

$$v_0(x, y) := \theta e^{-\pi^2 \frac{r^2}{\lambda^2}} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

- **Material properties:** Elastic subdomain: Granite | Acoustic subdomain: Water
- **Computational parameters:** SDIRK(3,4),  $n = 8$ ,  $l = 7$ ,  $k = 2$



**Fig. 12:** Left panel: Acoustic pressure (upper side) and elastic velocity norm (lower side) at time  $t = 0.375s$ . Right panel: Comparison to analytical solution (Gar6more).

## Propagation of an elastic pulse in sedimentary basin and atmosphere

■ **Material properties:**

- ▶ **Sedimentary basin:**  $\rho^S = 1200 \text{ kg.m}^{-3}$ ,  $c_p^S = 3400 \text{ m.s}^{-1}$ ,  $c_s = 1400 \text{ m.s}^{-1}$
- ▶ **Bedrock:**  $\rho^S = 5350 \text{ kg.m}^{-3}$ ,  $c_p^S = 3090 \text{ m.s}^{-1}$ ,  $c_s = 2570 \text{ m.s}^{-1}$
- ▶ **Air:**  $\rho^F = 1.292 \text{ kg.m}^{-3}$ ,  $c_p^F = 340 \text{ m.s}^{-1}$

## Propagation of an elastic pulse in sedimentary basin and atmosphere

■ **Material properties:**

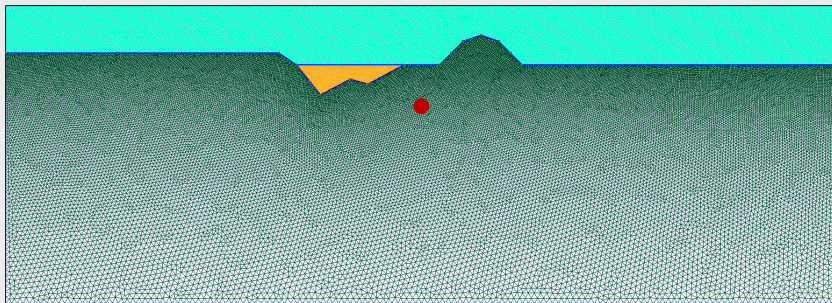
- ▶ **Sedimentary basin:**  $\rho^S = 1200 \text{ kg.m}^{-3}$ ,  $c_p^S = 3400 \text{ m.s}^{-1}$ ,  $c_s = 1400 \text{ m.s}^{-1}$
- ▶ **Bedrock:**  $\rho^S = 5350 \text{ kg.m}^{-3}$ ,  $c_p^S = 3090 \text{ m.s}^{-1}$ ,  $c_s = 2570 \text{ m.s}^{-1}$
- ▶ **Air:**  $\rho^F = 1.292 \text{ kg.m}^{-3}$ ,  $c_p^F = 340 \text{ m.s}^{-1}$

■ **Computational parameters:** SDIRK(3,4),  $k = 1$ ,  $\ell = 8$ ,  $n = 9$ ■ **Homogeneous Dirichlet boundary conditions**■ **Initial condition:** velocity Ricker wavelet centered at point  $(x_c, y_c) \in \Omega_{scs}$

## Propagation of an elastic pulse in sedimentary basin and atmosphere

■ **Material properties:**

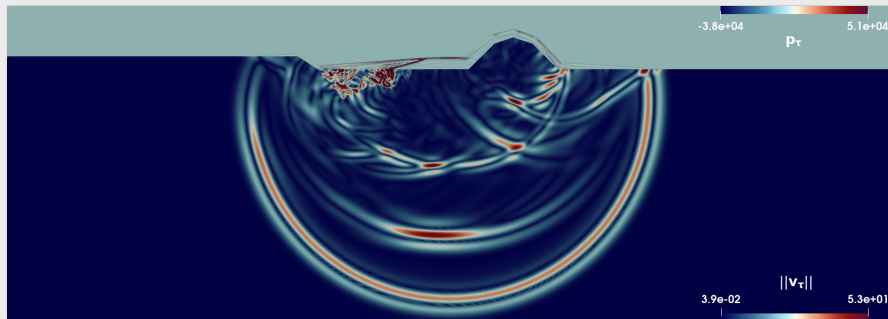
- ▶ **Sedimentary basin:**  $\rho^S = 1200 \text{ kg.m}^{-3}$ ,  $c_p^S = 3400 \text{ m.s}^{-1}$ ,  $c_s = 1400 \text{ m.s}^{-1}$
- ▶ **Bedrock:**  $\rho^S = 5350 \text{ kg.m}^{-3}$ ,  $c_p^S = 3090 \text{ m.s}^{-1}$ ,  $c_s = 2570 \text{ m.s}^{-1}$
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■ **Computational parameters:** SDIRK(3,4),  $k = 1$ ,  $\ell = 8$ ,  $n = 9$ ■ **Homogeneous Dirichlet boundary conditions**■ **Initial condition:** velocity Ricker wavelet centered at point  $(x_c, y_c) \in \Omega_{scs}$ 

**Fig. 13:** Mesh of sedimentary basin

## Propagation of elastic pulse in sedimentary basin and atmosphere

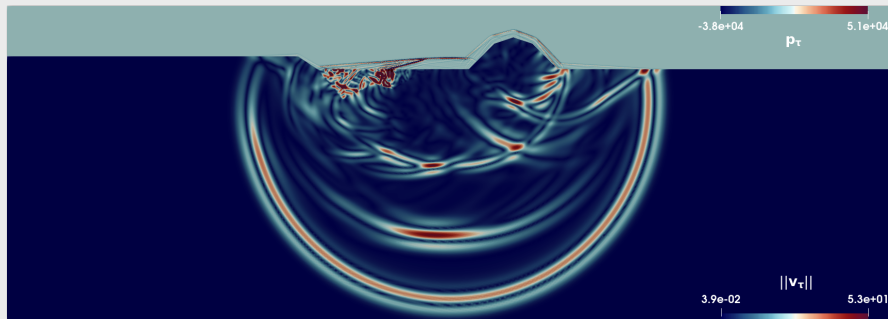
## ■ Energy transfer enhancement above sedimentary basin



**Fig. 14:** Propagation of elastic pulse in sedimentary basin and atmosphere

## Propagation of elastic pulse in sedimentary basin and atmosphere

## ■ Energy transfer enhancement above sedimentary basin



**Fig. 14:** Propagation of elastic pulse in sedimentary basin and atmosphere

Thank you for your attention !